

# Game-theoretic Modeling of Players' Ambiguities on External Factors

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## Abstract

We propose a game-theoretic framework that incorporates both incomplete information and general ambiguity attitudes on factors external to all players. Our starting point is players' preferences on payoff-distribution vectors, essentially mappings from states of the world to distributions of payoffs to be received by players. There are two ways in which equilibria for this preference game can be defined. When the preferences possess ever more features, we can gradually add ever more structures to the game. These include real-valued utility-like functions over payoff-distribution vectors, sets of probabilistic priors over states of the world, and eventually the traditional expected-utility framework involving one single prior. We establish equilibrium existence results and uncover relations between the two versions of equilibria. Players' ambiguity attitudes figure large in these results. Particular attention is paid to what we shall call the enterprising game, in which players exhibit ambiguity seeking attitudes while betting optimistically on the favorable resolution of ambiguities. The two solution concepts are unified at this game's pure equilibria, whose existence is guaranteed when strategic complementarities are present. The current framework can be applied to used-car sales involving retaliatory buyers, auctions involving ambiguity on competitors' assessments of item worths, and competitive pricing involving uncertain demand curves.

**Key words:** Incomplete Information; Preference Relation; Ambiguity; Strategic Complementarities; Auction; Competitive Pricing

# 1 Introduction

## 1.1 Our Motivation

In the traditional expected-utility approach to games involving incomplete information, such as Harsanyi's [22], players' types are used to model private messages they receive about the uncertain external environments. After observing his own type  $t_n$ , it is customary that player  $n$  will form a probabilistic understanding  $p_{n,t_n} \equiv (p_{n,t_n|t_{-n}})_{t_{-n} \in T_{-n}}$  about other players' types  $t_{-n} \in T_{-n}$ . Presented with others' strategies, he will strive to maximize his own expected utility, where expectation is taken with the aforementioned assessment  $p_{n,t_n}$ . To precisely describe the actual uncertainty, however, it often takes more than even the entire collection  $t \equiv (t_n)_{n \in N}$  of all players' types. This is where states of the world  $\omega$  will come into play.

A more detailed model might allow player  $n$ , after observing his own type  $t_n$ , to predict that the actual  $\omega$  will come from some subset  $\Omega_{n,t_n}$  of states of the world. Of course, it would be necessary that  $\Omega_{n,t_n}$  and  $\Omega_{n,t'_n}$  be non-overlapping when  $t_n \neq t'_n$ , and that  $\bigcup_{t_n \in T_n} \Omega_{n,t_n}$  for every player  $n$  be some common space  $\Omega$ . Indeed, such is essentially the information structure introduced by Aumann [4]. However, there seems to be no urgency in exploiting the “ $\omega$ -level” detail in the traditional approach, due to the “integration” of its effects.

Such is not necessarily the case when players' ambiguities have to be taken into account. Suppose all players adopt potentially random actions based on their types. When the actual  $\omega \in \Omega$  is eventually revealed, all players' types will be known: there is a unique  $t \equiv (t_n)_{n \in N}$  such that  $\omega \in \Omega_t \equiv \bigcap_{n \in N} \Omega_{n,t_n}$ . Then, every player will know the probabilistic distribution over his payoffs that he is supposed to experience. Of course, post decision, he will eventually experience just one single payoff. Now during the play, right after receiving his own type  $t_n$  but before knowing anything about others' types  $t_{-n}$  let alone the true  $\omega$ , player  $n$  should anticipate one payoff distribution say  $\pi(\omega)$  per  $\omega \in \Omega_{n,t_n} \equiv \bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n,t_{-n}}$ . He will certainly want to make the vector  $\pi \equiv (\pi(\omega) | \omega \in \Omega_{n,t_n})$  as likable to himself as possible. Each payoff-distribution vector  $\pi$  is essentially an act first proposed by Anscombe and Aumann [3].

A natural apparatus to express the “ $(n, t_n)$ ”-player's preferences is a strict preference relationship  $\succ_{n,t_n}$  on all payoff-distribution vectors, supposedly both irreflexive and transitive. With it, one can understand  $\pi \succ_{n,t_n} \pi'$  as the player-type pair liking  $\pi$  better than  $\pi'$  but not the other way around. The traditional expected-utility approach basically uses what we shall call a real-valued satisfaction function  $s_{n,t_n}$  on payoff-distribution vectors  $\pi$  to facilitate each  $(n, t_n)$ -player's preference relation:

$$\pi \succ_{n,t_n} \pi' \quad \text{if and only if} \quad s_{n,t_n}(\pi) > s_{n,t_n}(\pi'); \quad (1)$$

moreover,  $s_{n,t_n}$  is specially built from one single probabilistic prior  $\rho_{n,t_n}$  on the state space  $\Omega_{n,t_n}$  and a utility function  $u_{n,t_n}$  on payoffs, in the fashion of

$$s_{n,t_n}(\pi) = s_{n,t_n}^0(\pi, \rho_{n,t_n}) = \int_{\Omega_{n,t_n}} \left[ \int u_{n,t_n} \cdot d[\pi(\omega)] \right] \cdot \rho_{n,t_n}(d\omega). \quad (2)$$

This, at least in formality, amounts to  $s_{n,t_n}(\pi) = \int u_{n,t_n} \cdot d[\int_{\Omega_{n,t_n}} \pi(\omega) \cdot \rho_{n,t_n}(d\omega)]$  as well.

So it is as though the various  $\pi(\omega)$ -components are first averaged over using weights provided by the  $\rho_{n,t_n}$ -distribution; then the integrated payoff distribution or lottery is combined with the utility function  $u_{n,t_n}$  to generate the satisfaction level over the given payoff-distribution vector  $\pi$ . The linear treatment of the lottery on payoffs through the utility function  $u_{n,t_n}$  can be legitimized by von Neumann and Morgenstern's [40] axioms; the appropriateness of using a single prior  $\rho_{n,t_n}$  to assemble the lottery can be reasoned using Savage's [41] arguments. Both features have been used in the traditional modeling of not only incomplete-information but also normal-form games; see, e.g., Harsanyi [22] and Nash [38][39]. Of course, there is no need to single out the latter because they can be treated as special incomplete-information games in which every player has only one type.

Both linear treatments of payoff distributions and the latter's formations out of payoff-distribution vectors through the uses of single probabilistic priors have been criticized. First, Allais [2] challenged the notion that people use linear functionals over payoff distributions to reach decisions. Empirical studies in support of this contention can be found, e.g., in Camerer and Ho [8] and Wu and Gonzalez [50].

Then, Ellsberg [16] argued that decision makers (DMs) often do not even know the probabilities to be assigned to different states of the world. For instance, there are probably not enough data to estimate the chance of a new financial crisis to occur within the next two years; also, there has no precedent to be relied on to assess probabilities concerning the climate change due to human activities. Hence, to many situations the single-prior assumption on uncertain factors can be ill suited. Starting from Schmeidler [43], researchers resorted to tools like Choquet integration and capacities, i.e., non-additive probabilities, to help with single-agent decision making involving general ambiguity attitudes; see, e.g., Gilboa and Marinacci [20]. Under axioms associated with ambiguity aversion, Gilboa and Schmeidler [21] legitimized the worst-prior form to be taken by a DM. In this form,

$$s_{n,t_n}(\pi) = \inf_{\rho \in P_{n,t_n}} s_{n,t_n}^0(\pi, \rho), \quad (3)$$

where  $P_{n,t_n}$  is a set of prior distributions on the space  $\Omega_{n,t_n}$  and  $s_{n,t_n}^0$  is defined in (2).

## 1.2 Main Contributions

In the current strategic setting involving incomplete information, failure to account for players' diverse ambiguity attitudes could lead to weird predictions or dangerous prescriptions. In auctions, especially those involving works of art, offshore oilfields, or electromagnetic spectra, participants often do not know for sure the actual worths to themselves of the item being auctioned; very likely, they are also uncertain about the distributions their competitors assign to the item's worths; in addition, some may fear losing the object more than they regret about overpaying for it. How can a model capture these features then? The prevalent auction theory takes the traditional approach to incomplete-information games; hence, it is not able to model bidders' unconventional ambiguity attitudes.

We make an attempt at overcoming the above shortcoming by defining a more general preference game from the mere preference relations  $\succ_{n,t_n}$ , without imposing any structural requirement. Incidentally, a certain set of requirements led Anscombe and Aumann [3] to the simultaneous emergence of both the utility function  $u_{n,t_n}$  and the probabilistic prior  $\rho_{n,t_n}$  of the traditional form (2). But without help from any such structure, behavioral equilibria can already be defined. There is admittedly a growing game-theoretic literature on ambiguity considerations. Against this backdrop, this work still makes substantial contributions.

First, we propose a game-theoretic framework starting from players' preference relations on payoff-distribution vectors. This enables the incorporation of players' diverse ambiguity attitudes on external factors. Our emphasis here is not the consideration of preferences itself. In various strategic settings, this has been done by, e.g., Schmeidler [42], Mas-Colell [31], Shafer and Sonnenschein [44], and Khan and Sun [26]. It is preferences on *payoff-distribution vectors* that we want to stress. We believe such preferences provide more flexibility than those on actions, action distributions, integrated payoff distributions, or payoff vectors. This can probably be attested to by the inclusion of various existing models in the current framework.

Second, we give definitions to two prominent types of behavioral equilibria and establish their existence in various circumstances. Previously unknown relations between the two equilibrium notions are uncovered as well. The first, action-based interpretation leaves every player in control of his action whilst maintaining a long-term commitment to his portion of a behavioral equilibrium. The second, distribution-based interpretation ties every player's action to the outcome of a random device in a fashion consistent to his portion of an equilibrium. It will soon be clear that an action-based equilibrium assigns weights to only actions that leave no room for improvement by any other pure action; whereas, a distribution-based equilibrium leaves no room for improvement by any other distribution of actions. Since there

are “more” action distributions than pure actions to compete against, it is conceivable that distribution-based equilibria are in general “harder to come by” than action-based ones.

Third, we step into the less traveled ambiguity-seeking territory and make interesting findings. Since Ellsberg’s [16] pioneering work, most attention has been paid to ambiguity aversion as an alternative attitude to ambiguity neutrality. However, experiments involving human subjects showed that ambiguity seeking could be equally prevalent; see, e.g., Curley and Yates [10] and Charness, Karni, and Levin [9]. We also believe that optimistic assessments of uncertain gains is part of what drive people to participate in auctions, embark on exploratory journeys, and start new firms. Thus, the case opposite to that assuming (3) is equally if not more interesting. We call the corresponding game “enterprising” because each

$$s_{n,t_n}(\pi) = \sup_{\rho \in P_{n,t_n}} s_{n,t_n}^0(\pi, \rho), \quad (4)$$

so that players make optimistic bets on favorable resolutions of their ambiguities.

The action-distribution distinction turns out to be irrelevant for the enterprising game’s pure equilibria, given that they exist. When equipped with strategic complementarity features, the game can be shown to possess not only pure equilibria, but also those with monotone trends with respect to players’ types as well as external conditions. These results can be considered as extensions of those achieved for the traditional counterpart as laid out in van Zandt and Vives [52]. As normal-form games are incomplete-information games with singleton type spaces, the results also generalize those that appeared in traditional supermodular games studied by Topkis [47], Milgrom and Roberts [33], and Vives [49].

Finally, our findings are applicable to settings like used-car sales involving retaliatory buyers, auctions involving ambiguity on competitors’ assessments of item worths, and competitive pricing involving uncertain demand curves.

### 1.3 Outline of Results

Under mild conditions, we show that action-based equilibria always exist. When the preferences  $\succ_{n,t_n}$  connote ambiguity aversion, distribution-based ones will come into being as well; see Theorem 1. When the preferences are representable by real-valued functions  $s_{n,t_n}$  satisfying (1), our game is specialized to the so-called satisfaction kind. For this game, action-based equilibria will exist in general and so will distribution-based equilibria when the  $s_{n,t_n}$ ’s are quasi-concave. When there is a set  $P_{n,t_n}$  of prior distributions on the state space  $\Omega_{n,t_n}$ , so that each  $s_{n,t_n}$  takes Gilboa and Schmeidler’s [21] form (3), we shall obtain the so-called

alarmists' game. In it, players express aversions to ambiguities. Due to concavity of the  $s_{n,t_n}$ 's, the game can be shown to have both action- and distribution-based equilibria.

For the preference game, rudimentary understandings on relations between the action- and distribution-based equilibria can be formed. Our message will become considerably sharper for the satisfaction game. For it, we can conclude that distribution-based equilibria will be action-based ones when players are ambiguity-seeking and the two types will be identical when players are ambiguity-neutral; see Theorem 2. Consequently, as might have been suspected, the distinction between the two versions of equilibria will cease to matter for the traditional expected-utility game. Our derivation relies on concepts like continuous kernels and their integrations, as well as intermediate results like Lemma 1 that might be of value to other situations. When we focus on pure equilibria, we again confirm the earlier "comparative rarity" observation by showing that any pure distribution-based equilibrium must also be a pure action-based one; see Theorem 3.

Our attention then shifts to the enterprising game in which players demonstrate ambiguity-seeking traits. As a special satisfaction game with convex  $s_{n,t_n}$  functions, any distribution-based equilibrium of this game is necessarily an action-based one. When confined to pure strategies, we also have the equivalence between the two types of equilibria; see Theorem 4. One technical result involved in its proof is Lemma 2. It is an extension of a well known finite-dimensional property, stating that the maximum of a convex function over a convex region in  $\mathbb{R}^d$  for some  $d = 1, 2, \dots$  can always be achieved at extreme points.

Of special interest is the case where (i) each  $\Omega_t = \{t\} \times \tilde{\Omega}$  for some common state space  $\tilde{\Omega}$  and (ii) all ambiguities of an  $(n, t_n)$ -player are on  $\tilde{\Omega}$  rather than other players' types  $t_{-n}$ . This reflects the situation where players can form subjective probabilities  $p_{n,t_n|t_{-n}}$  on their opponents' types much like in the traditional game, but with extra ambiguities on other external factors. To this case, we can extend the traditional analysis of games possessing strategic complementarities and obtain the existence of monotone pure equilibria as well as their monotone comparative statics properties.

In particular, Theorem 5 demonstrates that pure equilibria where prescribed actions monotonically increase with players' types are in existence. A technical result enabling the theorem's proof is Lemma 3 on the preservation of increasing differences under maximization, much like a well known one about the preservation of supermodularity under maximization. Also, Theorem 6 shows that equilibria can evolve monotonically with a game-defining parameter when the latter guides the game to transform in a certain monotone fashion. Our derivations borrowed ideas from works dealing with subjects like lattices and submodularity,

including Milgrom and Shannon [34], Zhou [53], Topkis [48], and Yang and Qi [51].

In the rest of the paper, we discuss existing game-theoretic literature with ambiguity considerations in Section 2. Then, we give a general formulation in Section 3, make detailed analysis in Section 4, delve into various special cases in Section 5, and establish relationships between the two equilibrium concepts in Section 6. We focus on a special enterprising game with strategic-complementarity features and its monotone pure equilibria in Section 7. Applications are postulated in Section 8; finally, the paper is concluded in Section 9.

## 2 Literature Survey

Normal-form games incorporating general ambiguity attitudes have been studied. Dow and Werlang [14] used convex capacities to model players' beliefs about opponents' behaviors and arrived at equilibrium belief profiles. Eichberger and Kelsey [15] extended the study to situations involving  $n \geq 3$  players and identified players' confidence degrees for equilibrium parametrization purposes. Marinacci [30], on the other hand, gave more flexible definitions to players' vaguenesses on their beliefs, which could then be used in comparative statics studies. Klibanoff [25] and Lo [28] adopted Gilboa and Schmeidler's [21] notion of ambiguity aversion and used convex and closed sets of probabilistic priors on products of other players' mixed strategies, reducible to those on their pure actions, as the basis on which players make decisions. Epstein [17] let players be ambiguous about opponents' pure strategies as well as their ambiguity attitudes, and studied the iterated elimination of dominated strategies.

Players in the above games were allowed to have qualms about opponents' behaviors. We, like some studies of incomplete-information games involving general ambiguity attitudes, focus on the complementary situation where players have vagueness about factors external to all of them. We argue for merits of the ambiguity-on-external-factor rather than ambiguity-on-opponent-behavior consideration as follows. First, as shown momentarily, mixed strategies chosen by players are often enforceable. Second, uncertainties about the state of the world can pose a much bigger problem than those about other players' behaviors. Think of a Stag Hunt game where each participant has only to choose between *cooperate* and *defect*, and yet there are millions of combinations in numbers, sizes, and speeds of the stags and hares on the hunting ground, as well as other factors like temperature and wind. Third, no longer having to model players' behaviors through non-probabilistic means, we can apply conventional tools built on countably additive probabilities to our analysis.

We want to emphasize, as has been brought up in Azrieli and Teper [5], that in a gen-

uine incomplete-information setting where some players do have multiple types, uncertainty about opponents' types will still indirectly lead to uncertainty about their behaviors. Incidentally, Bade [6] introduced behavioral ambiguity by allowing players to base their actions on uncontrollable factors that are not observable by opponents.

Regarding the verifiability of mixed strategies pronounced by players, there seem to be at least two plausible solutions. First, when the game merely reflects one encounter in many repeated interactions, we propose that each player takes a frequentist approach to the compliance of his proclaimed mixed strategy. Thus, he is at almost total control of his own action in each play, but has to maintain agreed-upon frequencies to various actions in the long run. Second, it is possible that every player is given a random number generator whose output is private knowledge in-game but public knowledge post-game, and the player has to act according to an agreed-upon mapping from the random device's output to his action. We call the first perspective action-based because players get to choose their actual actions and the second distribution-based because players need to decide on distributions of actions before they start to act. Equilibrium concepts in Dow and Werlang [14] and Marinacci [30] are of the action-based variety; whereas, those in Klibanoff [25] and Lo [28] are of the distribution-based variety. Also, Kajii and Ui [24] called the first kind "equilibria in beliefs" and the second kind "mixed equilibria".

There has always been the interim-*ex ante* distinction in the traditional incomplete-information game. In this regard, we can be said to have taken the interim approach, letting players express their preferences one type a time. Since player  $n$  has to make up his mind only after observing his own type  $t_n$ , there seems to be no reason why he should not strive for the best for each realized type. The *ex ante* approach, on the other hand, would prompt the study of players' preferences  $\succ_n$  on payoff-distribution "super" vectors  $\pi \equiv (\pi_{t_n})_{t_n \in T_n}$  in which every  $\pi_{t_n}$  is the erstwhile payoff-distribution vector. Due to dynamic consistency concerns, this alternative appears to be more complicated and yet less urgent.

Among works on incomplete-information games involving general ambiguity attitudes, we note that Epstein and Wang [18] used preference relations over acts to express ambiguity attitudes, and also allowed ambiguities over opponents' preferences. This setup gave rise to infinite sequences of preferences over preferences, much like Mertens and Zamir's [32] sequences of beliefs over beliefs. Under reasonable assumptions about allowable preferences, authors justified the emergence of those largest necessary type spaces that contain players' personal characteristics. Ahn [1] and Di Tillio [13] worked along a similar line, with the former modeling ambiguities using sets of beliefs and the latter imposing less restrictions on



preferences but more on payoff and state spaces.

In our current study, exogenous factors  $\omega$  within the state space  $\Omega$  contain no information on either opponents' behaviors or their ambiguity attitudes. Each of player  $n$ 's types  $t_n$  is a private message he receives about the actual external factor. The  $(n, t_n)$ -player is uncertain which  $\omega \in \Omega_{n, t_n}$  has been realized. Consequently, he is uncertain about the actual opponent-type profile  $t_{-n} \in T_{-n}$  because he does not know which  $\Omega_{t_n, t_{-n}} \subseteq \Omega_{n, t_n}$  the  $\omega$  is in. The various preference relations  $\succ_{n, t_n}$  on “ $|\Omega_{n, t_n}|$ -dimensional” payoff-distribution vectors reflect players' ambiguity attitudes which might be under the sway of messages they receive. Like their behaviors which as we have emphasized are observable and enforceable, players' preferences are assumed to be commonly known.

In real life, these entities might simply translate into a few choices on players' personalities. For instance, it is possible that one player be labeled “mildly conservative all the time” while another “fairly reckless when knowing that the stake is high”. Just allowing ambiguities on factors external to all players is applicable enough to a wide variety of practically relevant cases. Take, for example, an auction of the exploration right to an offshore oil field. All bidders, being major petrochemical firms, probably know each other well through past interactions. The major uncertainty then stems from the potential of the field itself, which is also responsible for all the private readings delivered to the bidding oil majors.

Some works, like ours, dealt with equilibrium existence issues. Kajii and Ui [24] effectively studied the alarmists' game, albeit with finite action and state spaces. They showed the existence of both action- and distribution-based equilibria; see, respectively, their Propositions 2 and 1. Moreover, Azrieli and Teper [5] treated what might be considered a special satisfaction game. In its interim version, payoff-distribution vectors  $\pi$  are first turned into expected-utility vectors  $\int u_{n, t_n} \cdot d\pi \equiv (\int u_{n, t_n} \cdot d[\pi(\omega)] | \omega \in \Omega_{n, t_n})$  using utility functions  $u_{n, t_n}$ . The latter are then assessed using functionals say  $j_{n, t_n}$ . So for this game,

$$s_{n, t_n}(\pi) = j_{n, t_n}(\int u_{n, t_n} \cdot d\pi). \quad (5)$$

We can get back to the traditional case of (2) when  $j_{n, t_n}$  is linear in the sense that  $j_{n, t_n}(f) = \int_{\Omega_{n, t_n}} f(\omega) \cdot \rho_{n, t_n}(d\omega)$  for some single prior  $\rho_{n, t_n}$ . For this particular satisfaction game, authors showed that quasi-concavity of the  $j_{n, t_n}$ 's would lead to the existence of distribution-based equilibria; see their Definition 2 which allows a player to maximize his action distribution instead of letting him fill the distribution's support with optimal pure actions.

Finally, we note that ambiguity considerations have been brought to studies of auctions; see, e.g., Lo [29] and Bose, Ozdenoren, and Pape [7].

### 3 General Formulation

#### 3.1 Game Primitives

Given space  $X$  with metric  $d_X$ , we use  $\mathcal{B}(X)$  for its Borel  $\sigma$ -field and  $\mathcal{P}(X)$  for the space of probabilities defined on the measurable space  $(X, \mathcal{B}(X))$ . The space  $\mathcal{P}(X)$  is endowed with the Prokhorov metric  $\psi_X$ , which also induces weak convergence. It will be separable when  $X$  is. When the latter is compact,  $\mathcal{P}(X)$  will be so too. Given metric spaces  $X$  and  $Y$ , let  $\mathcal{C}(X, Y)$  be the space of continuous mappings from  $X$  to  $Y$ . Its members will be uniformly continuous when  $X$  is compact; they will further be bounded when  $Y$  is the real line  $\mathbb{R}$ .

We let the finite  $N \equiv \{1, \dots, \bar{n}\}$  be the set of players. Each player  $n \in N$  is associated with a finite type space  $T_n \equiv \{1, \dots, \bar{t}_n\}$ . For convenience, we call player  $n$  with type  $t_n$  the  $(n, t_n)$ -player. This player's action comes from some metric space  $A_{n, t_n}$ . Let  $T \equiv \prod_{n \in N} T_n$  be the space of type profiles and for each such profile  $t \equiv (t_n)_{n \in N} \in T$ , let  $A_t \equiv \prod_{n \in N} A_{n, t_n}$  be the space of allowable action profiles under  $t$ .

Suppose metric space  $\Omega$  hosts states of the world. Given  $n \in N$ , let  $(\Omega_{n, t_n})_{t_n \in T_n}$  be a partition of  $\Omega$ , with each  $\Omega_{n, t_n}$  containing all states of the world that correspond to player  $n$ 's type  $t_n$ . Even when player  $n$  knows his type to be a certain  $t_n$ , he should anticipate the state of the world  $\omega$  to come from anywhere in  $\Omega_{n, t_n}$ . Given  $t \equiv (t_n)_{n \in N} \in T$ , use  $\Omega_t \equiv \bigcap_{n \in N} \Omega_{n, t_n}$  for the set hosting all states of the world that correspond to each player  $n$  his type  $t_n$ . Spaces concerning the states of the world or in our own words, external factors, can be “averaged away” from the modeler's view if the traditional approach is taken. Here, with general ambiguity attitudes being considered, they will not. The idea of using partitions of event sets to reflect information structure dates as far back to as Aumann [4].

After introducing players, types, actions, and states of the world, we now turn to payoffs. For  $n \in N$  and  $t \in T$ , let there be Borel-measurable functions  $r_{n, t} \equiv r_{n, t_n, t_{-n}}$  from  $A_t \times \Omega_t$  to some metric space  $R_{n, t_n}$ , so that each  $r_{n, t_n, t_{-n}}(a, \omega)$  stands for the payoff to player  $n$  under type profile  $t \equiv (t_n, t_{-n})$ , pure action profile  $a \equiv (a_n)_{n \in N} \in A_t$ , and state of the world  $\omega \in \Omega_t$ . Note that the payoff spaces  $R_{n, t_n}$  do not have to be one-dimensional real sets.

Each player  $n$ , when seeing his type  $t_n$ , will be able to tell that the actual realization  $\omega$  is in  $\Omega_{n, t_n}$ ; however, nothing else, including opponents' types, can be determined. During the game's play where opponents mete out their behavioral strategies while the player himself may or may not randomize on actions, he will face choices on payoff-distribution vectors of the form  $\pi \equiv (\pi(\omega) | \omega \in \Omega_{n, t_n})$ , where each component  $\pi(\omega)$  is a member of  $\mathcal{P}(R_{n, t_n})$ , the space of payoff distributions.

It will become clear that Harsanyi's [22] approach and by specialization, Nash's [38][39] as well, represent a very special view on how players should rank the payoff-distribution vectors  $\pi$ . Here, we make generalizations. First, we just associate each  $(n, t_n)$ -player with a strict preference relation  $\succ_{n, t_n}$  on the space  $(\mathcal{P}(R_{n, t_n}))^{\Omega_{n, t_n}}$  of such vectors. The relation is merely required to be irreflexive and transitive, to the effect that

- (I)  $\pi \not\succ_{n, t_n} \pi$  for any  $\pi \in (\mathcal{P}(R_{n, t_n}))^{\Omega_{n, t_n}}$ ;
- (II)  $\pi \succ_{n, t_n} \pi''$  whenever  $\pi \succ_{n, t_n} \pi'$  and  $\pi' \succ_{n, t_n} \pi''$ .

For example, it might be that  $\Omega_{n, t_n} = \{\text{cold weater, hot weather}\}$  and  $R_{n, t_n} = \{(n_I, n_S) | n_I, n_S = 0, 1, \dots\}$  with  $n_I$  standing for the number of ice creams and  $n_T$  the number of cups of hot tea. Then, one  $\succ_{n, t_n}$  might dictate that “having one cup of hot tea for sure when it is cold and having one ice cream for sure when it is hot” is strictly preferred to “having one ice cream and one cup of hot tea both with 50% chances regardless whether it is cold or hot”, which is in turn strictly preferred to “having one ice cream for sure when it is cold and having one cup of hot tea for sure when it is hot”.

We focus on the preference game  $\Gamma \equiv (N, (T_n)_{n \in N}, (A_{n, t_n})_{n \in N, t_n \in T_n}, \Omega, (\Omega_{n, t_n})_{n \in N, t_n \in T_n}, (R_{n, t_n})_{n \in N, t_n \in T_n}, (r_{n, t})_{n \in N, t \in T}, (\succ_{n, t_n})_{n \in N, t_n \in T_n})$ . To recap,  $N$  is the set of players, each  $T_n$  is player  $n$ 's type space, and each  $A_{n, t_n}$  is the action space of the  $(n, t_n)$ -player; also,  $\Omega$  is the space for states of the world and each  $\Omega_{n, t_n}$  contains states of the world that lead to player  $n$ 's type  $t_n$ ; finally, each  $R_{n, t_n}$  is the payoff space of the  $(n, t_n)$ -player, each  $r_{n, t} \equiv r_{n, t_n, t_{-n}}$  is that player's payoff function from  $A_t \times \Omega_t$  to  $R_{n, t_n}$  when opponents' type profile happens to be  $t_{-n}$ , and each  $\succ_{n, t_n}$  is the preference relation adopted by the  $(n, t)$ -player.

### 3.2 Payoff-distribution Vectors

Every player  $n$ 's behavioral strategy  $\delta_n$  can be understood as the vector  $(\delta_{n, t_n})_{t_n \in T_n}$ , where each component  $\delta_{n, t_n} \in \Delta_{n, t_n} \equiv \mathcal{P}(A_{n, t_n})$  is a probability distribution over actions in  $A_{n, t_n}$ . That is,  $\delta_n \in \Delta_n \equiv \prod_{t_n \in T_n} \Delta_{n, t_n}$ . This way,  $\delta_n$  offers a plan for player  $n$  on what potentially randomized action to take under each type realization  $t_n$ .

Let  $\Delta \equiv \prod_{n \in N} \Delta_n$  be the space of all behavioral-strategy profiles covering all players. To the  $(n, t_n)$ -player, opponents' type profile  $t_{-n} \equiv (t_m)_{m \neq n}$  may be anything from  $T_{-n} \equiv \prod_{m \neq n} T_m$  and their behavioral-strategy profile  $\delta_{-n} \equiv (\delta_m)_{m \neq n}$  may be anything from  $\Delta_{-n} \equiv \prod_{m \neq n} \Delta_m$ . At each fixed  $t_{-n} \in T_{-n}$ , it is the  $\delta_{-n, t_{-n}} \equiv (\delta_{m, t_m})_{m \neq n}$ -portion of  $\delta_{-n}$  that will materialize, and the state of the world  $\omega$  must be from  $\Omega_{t_n, t_{-n}}$ . Note that  $(\Omega_{t_n, t_{-n}})_{t_{-n} \in T_{-n}}$  forms a partition of  $\Omega_{n, t_n}$ . In the following, we will take the liberty to use notation like  $a(\text{ or } \delta)_{-n, t_{-n}} \equiv (a(\text{ or } \delta)_{m, t_m})_{m \neq n}$  and  $A(\text{ or } \Delta)_{-n, t_{-n}} \equiv \prod_{m \neq n} A(\text{ or } \Delta)_{m, t_m}$ .

As noted, there can be two ways in which payoff-distribution vectors can be formed during the play of a game where players' random actions are verifiable and yet ambiguities exist on external factors. The action-based case will emerge when each player has almost a free reign on the actions to take except with the long-term goal of playing out a given randomized strategy; meanwhile, the distribution-based case will arise when players use exogenously generated random numbers to map out their chosen random strategies.

From the action-based perspective, the  $(n, t_n)$ -player is in total command of his own action whilst anticipating random actions from other players. Then, under his pure action  $a_{n,t_n} \in A_{n,t_n}$ , opponent behavioral-strategy profile  $\delta_{-n} \equiv (\delta_{-n,t_{-n}})_{t_{-n} \in T_{-n}} \equiv (\delta_{m,t_m})_{m \neq n, t_m \in T_m} \in \Delta_{-n}$ , opponent type profile  $t_{-n} \in T_{-n}$ , and state of the world  $\omega \in \Omega_{t_n, t_{-n}}$ , the player will expect the payoff distribution

$$\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega) = \left( \prod_{m \neq n} \delta_{m,t_m} \right) \cdot (r_{n,t_n,t_{-n}}(a_{n,t_n}, \cdot, \omega))^{-1} \in \mathcal{P}(R_{n,t_n}). \quad (6)$$

For any  $R'_{n,t_n} \in \mathcal{B}(R_{n,t_n})$ , the probability  $[\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega)](R'_{n,t_n})$  equals

$$\begin{aligned} & \int_{A_{-n,t_{-n}}} \mathbf{1}(\{a_{-n,t_{-n}} \text{ with } r_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}}, \omega) \in R'_{n,t_n}\}) \cdot [\prod_{m \neq n} \delta_{m,t_m}](da_{-n,t_{-n}}) \\ &= (\prod_{m \neq n} \delta_{m,t_m})(\{a_{-n,t_{-n}} \in A_{-n,t_{-n}} | r_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}}, \omega) \in R'_{n,t_n}\}), \end{aligned} \quad (7)$$

where  $\mathbf{1}(\cdot)$  stands for the indicator function. The above reflects how opponents' random actions result with the current player's random payoff distribution. Assumptions to be made in Section 4.1 will ensure that all operations in this and the ensuing Section 3.3 are legitimate. Had player  $n$  known his opponents' type profile  $t_{-n} \in T_{-n}$ , he would have anticipated the “ $|\Omega_{t_n, t_{-n}}|$ -dimensional” vector

$$\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}) \equiv (\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega) | \omega \in \Omega_{t_n, t_{-n}}). \quad (8)$$

However, the player is unaware of opponents' actual type profile. So he should contemplate on the “ $|\Omega_{n,t_n}|$ -dimensional” vector that is patched up from the vectors defined in (8):

$$\begin{aligned} (\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}))_{t_{-n} \in T_{-n}} &= ((\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega) | \omega \in \Omega_{t_n, t_{-n}}))_{t_{-n} \in T_{-n}} \\ &= (\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega) | \omega \in \Omega_{n,t_n}), \end{aligned} \quad (9)$$

where the second equality comes from  $\Omega_{n,t_n} = \bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n, t_{-n}}$ . The resulting payoff-distribution vector  $\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})$  is a member of  $(\mathcal{P}(R_{n,t_n}))^{\bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n, t_{-n}}} \equiv (\mathcal{P}(R_{n,t_n}))^{\Omega_{n,t_n}}$ .

From the distribution-based perspective, the  $(n, t_n)$ -player is in control of his behavioral strategy. When he is committed to some  $\delta_{n,t_n} \in \Delta_{n,t_n}$  while other players have adopted

behavioral-strategy profile  $\delta_{-n} \in \Delta_{-n}$ , the player should, under opponent type profile  $t_{-n} \in T_{-n}$  and state of the world  $\omega \in \Omega_{t_n, t_{-n}}$ , anticipate the additionally mixed distribution

$$\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega) = (\delta_{n, t_n} \times \prod_{m \neq n} \delta_{m, t_m}) \cdot (r_{n, t_n, t_{-n}}(\cdot, \cdot, \omega))^{-1} \in \mathcal{P}(R_{n, t_n}). \quad (10)$$

In view of (6) and (7), the above (10) could also be understood as

$$\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega) = \int_{A_{n, t_n}} \pi_{n, t_n, t_{-n}}^a(a_{n, t_n}, \delta_{-n, t_{-n}}, \omega) \cdot \delta_{n, t_n}(da_{n, t_n}), \quad (11)$$

in the sense that, for any  $R'_{n, t_n} \in \mathcal{B}(R_{n, t_n})$ ,

$$[\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega)](R'_{n, t_n}) = \int_{A_{n, t_n}} [\pi_{n, t_n, t_{-n}}^a(a_{n, t_n}, \delta_{-n, t_{-n}}, \omega)](R'_{n, t_n}) \cdot \delta_{n, t_n}(da_{n, t_n}). \quad (12)$$

This just means that the payoff distribution under the  $(n, t_n)$ -player's behavioral strategy  $\delta_{n, t_n}$  is a mixture of the payoff distributions under the player's pure strategies. Now, let vector  $\pi_{n, t_n}^d(\delta_{n, t_n}, \delta_{-n})$  be equated to

$$\begin{aligned} (\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}))_{t_{-n} \in T_{-n}} &\equiv ((\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega) | \omega \in \Omega_{t_n, t_{-n}}))_{t_{-n} \in T_{-n}} \\ &= (\pi_{n, t_n, t_{-n}}^d(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega) | \omega \in \Omega_{n, t_n}), \end{aligned} \quad (13)$$

i.e., the vector of all potential payoff distributions under opponent type profiles  $t_{-n} \in T_{-n}$  and states of the world  $\omega \in \Omega_{t_n, t_{-n}}$ . It is again a member of  $(\mathcal{P}(R_{n, t_n}))^{\Omega_{n, t_n}}$ .

### 3.3 Equilibrium Definitions

The two perspectives lead to two equilibrium notions. They are potentially different in cases with general ambiguity attitudes. In the action-based case, each player  $n$  should respond to any given opponent strategy profile  $\delta_{-n}$  by choosing strategy  $\delta_{n, t_n}$  for each type realization  $t_n$  that gives no chance to any action  $a_{n, t_n}$  whose corresponding vector  $\pi_{n, t_n}^a(a_{n, t_n}, \delta_{-n})$  defined at (9) could be less preferential than that of any other action. In the distribution-based case, the player should respond to the same by choosing strategy  $\delta_{n, t_n}$  at each  $t_n$  so that the corresponding vector  $\pi_{n, t_n}^d(\delta_{n, t_n}, \delta_{-n})$  defined at (13) is not less preferential than that of any other mixed strategy.

Start from the action-based perspective. For each player  $n \in N$ , type  $t_n \in T_n$ , and opponent strategy profile  $\delta_{-n} \in \Delta_{-n}$ , let  $\hat{A}_{n, t_n}^a(\delta_{-n})$  be the set of actions  $a_{n, t_n}$  that render the vector  $\pi_{n, t_n}^a(a_{n, t_n}, \delta_{-n})$  as defined in (9)  $\succ_{n, t_n}$ -maximal:

$$\hat{A}_{n, t_n}^a(\delta_{-n}) = \{a_{n, t_n} \in A_{n, t_n} | \pi_{n, t_n}^a(a'_{n, t_n}, \delta_{-n}) \not\succ_{n, t_n} \pi_{n, t_n}^a(a_{n, t_n}, \delta_{-n}) \ \forall a'_{n, t_n} \in A_{n, t_n}\}. \quad (14)$$

For any  $n \in N$  and  $t_n \in T_n$ , let best-response correspondence  $\hat{B}_{n,t_n}^a : \Delta_{-n} \rightrightarrows \Delta_{n,t_n}$  be such that, for any opponent strategy profile  $\delta_{-n} \in \Delta_{-n}$ ,

$$\hat{B}_{n,t_n}^a(\delta_{-n}) = \{\delta_{n,t_n} \in \Delta_{n,t_n} | \delta_{n,t_n}(\hat{A}_{n,t_n}^a(\delta_{-n})) = 1\}, \quad (15)$$

where  $\hat{A}_{n,t_n}^a(\delta_{-n})$  has just been defined in (14). Thus,  $\delta_{n,t_n}$  will be considered one of the  $(n, t_n)$ -player's best responses to  $\delta_{-n}$  when its support is made up of those  $a_{n,t_n}$ 's that render  $\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) \succ_{n,t_n}$ -maximal among all  $\pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n})$ 's.

Now define correspondence  $\hat{B}^a : \Delta \rightrightarrows \Delta$  from strategy profiles to themselves, so that

$$\delta' \in \hat{B}^a(\delta) \text{ if and only if } \delta'_{n,t_n} \in \hat{B}_{n,t_n}^a(\delta_{-n}) \text{ for any } n \in N \text{ and } t_n \in T_n. \quad (16)$$

A behavioral-strategy profile  $\delta \equiv (\delta_{n,t_n})_{n \in N, t_n \in T_n} \in \Delta \equiv \prod_{n \in N} \prod_{t_n \in T_n} \Delta_{n,t_n}$  will be considered an action-based equilibrium of  $\Gamma$  if  $\delta \in \hat{B}^a(\delta)$ . For convenience, we use  $\mathcal{E}^a \subseteq \Delta$  to denote the set of all such equilibria.

Now move on to the distribution-based perspective. For any player  $n \in N$  and type  $t_n \in T_n$ , define best-response correspondence  $\hat{B}_{n,t_n}^d : \Delta_{-n} \rightrightarrows \Delta_{n,t_n}$  so that, for any opponent strategy profile  $\delta_{-n} \in \Delta_{-n}$ ,

$$\hat{B}_{n,t_n}^d(\delta_{-n}) = \{\delta_{n,t_n} \in \Delta_{n,t_n} | \pi_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}) \not\succ_{n,t_n} \pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}) \ \forall \delta'_{n,t_n} \in \Delta_{n,t_n}\}. \quad (17)$$

Here, a  $\delta_{n,t_n}$  will be considered one of the  $(n, t_n)$ -player's best responses to  $\delta_{-n}$  when  $\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})$  is  $\succ_{n,t_n}$ -maximal among all  $\pi_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n})$ 's.

Now define correspondence  $\hat{B}^d : \Delta \rightrightarrows \Delta$  from strategy profiles to themselves, so that

$$\delta' \in \hat{B}^d(\delta) \text{ if and only if } \delta'_{n,t_n} \in \hat{B}_{n,t_n}^d(\delta_{-n}) \text{ for any } n \in N \text{ and } t_n \in T_n. \quad (18)$$

A behavioral-strategy profile  $\delta \in \Delta$  will be considered a distribution-based equilibrium of  $\Gamma$  if  $\delta \in \hat{B}^d(\delta)$ . For convenience, we use  $\mathcal{E}^d \subseteq \Delta$  to denote the set of all such equilibria.

When every type space  $T_n$  is a singleton, the game  $\Gamma$  will be normal-form. Then, all the state spaces  $\Omega_t$  and  $\Omega_{n,t_n}$  will be equatable to  $\Omega$ ; hence, all payoff-distribution vectors will be of the same length. Certainly, no separate treatment is needed for this special case.

## 4 Existence of Equilibria

### 4.1 Compactness and Continuity

Let us provide conditions under which the equilibrium sets  $\mathcal{E}^a$  and  $\mathcal{E}^d$  will be nonempty. We first make the following assumptions related to compactness and continuity.

**Assumption 1** For any  $n \in N$  and  $t_n \in T_n$ , the action space  $A_{n,t_n}$  is compact.

**Assumption 2** The state space  $\Omega$  is compact and for any type profile  $t \equiv (t_n)_{n \in N} \in T \equiv \prod_{n \in N} T_n$ , the subset  $\Omega_t \equiv \bigcap_{n \in N} \Omega_{n,t_n}$  is closed and hence compact.

**Assumption 3** For any  $n \in N$  and  $t_n \in T_n$ , the payoff space  $R_{n,t_n}$  is compact.

**Assumption 4** For any player  $n \in N$  and type profile  $t \in T$ , the payoff function  $r_{n,t}$  from  $A_t \times \Omega_t \equiv \prod_{n \in N} A_{n,t_n} \times \bigcap_{n \in N} \Omega_{n,t_n}$  to  $R_{n,t_n}$  is continuous.

Assumptions 1 to 4 are all quite routine. The only exception might be Assumption 2, which requires that all the disjoint spaces  $\Omega_t$  be closed and hence “mutually distinguishable”. But this is trivially satisfied by the case where  $\Omega_t = \{t\} \times \tilde{\Omega}$  for some common space  $\tilde{\Omega}$ .

By Assumption 1, each  $A_t \equiv \prod_{n \in N} A_{n,t_n}$  is compact. Also, all the action-distribution spaces  $\Delta_{n,t_n} \equiv \mathcal{P}(A_{n,t_n})$ ,  $\Delta_{-n,t_{-n}} \equiv \prod_{m \neq n} \Delta_{m,t_m}$ ,  $\Delta_n \equiv \prod_{t_n \in T_n} \Delta_{n,t_n}$ ,  $\Delta_{-n} \equiv \prod_{m \neq n} \Delta_m$ , and  $\Delta \equiv \prod_{n \in N} \Delta_n$  will be compact. By Assumption 2, each  $\Omega_{n,t_n} \equiv \bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n,t_{-n}}$  is compact. Consequently, all the state-distribution spaces  $\mathcal{P}(\Omega_t)$ ,  $\mathcal{P}(\Omega_{n,t_n})$ , and  $\mathcal{P}(\Omega)$  will be compact. With Assumption 3, we have the compactness of the payoff-distribution space  $\mathcal{P}(R_{n,t_n})$ . Note that the payoff functions  $r_{n,t}$  are defined on compact domains  $A_t \times \Omega_t$ . Due also to the continuity stated in Assumption 4, we can obtain the continuity of both  $\pi_{n,t_n,t_{-n}}^a$  defined in (6) and  $\pi_{n,t_n,t_{-n}}^d$  defined in (10).

**Proposition 1** For any  $n \in N$ ,  $t_n \in T_n$ , and  $t_{-n} \in T_{-n}$ , the function  $\pi_{n,t_n,t_{-n}}^a$  defined at (6) from  $A_{n,t_n} \times \Delta_{-n,t_{-n}} \times \Omega_{t_n,t_{-n}}$  to the payoff-distribution space  $\mathcal{P}(R_{n,t_n})$  is continuous; also, the function  $\pi_{n,t_n,t_{-n}}^d$  defined at (10) from  $\delta_{n,t_n} \times \Delta_{-n,t_{-n}} \times \Omega_{t_n,t_{-n}}$  to  $\mathcal{P}(R_{n,t_n})$  is continuous.

Due to the compactness of all involved spaces, we can obtain from Proposition 1 the uniform continuity of  $\pi_{n,t_n,t_{-n}}^a$  and  $\pi_{n,t_n,t_{-n}}^d$ . So instead of  $(\mathcal{P}(R_{n,t_n}))^{\Omega_{t_n,t_{-n}}}$ , we can restrict the vectors  $\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}})$  and  $\pi_{n,t_n,t_{-n}}^d(\delta_{n,t_n}, \delta_{-n,t_{-n}})$  defined in (8) and (13), respectively, to the smaller  $\mathcal{C}(\Omega_{t_n,t_{-n}}, \mathcal{P}(R_{n,t_n}))$ , the space of all uniformly continuous mappings from  $\Omega_{t_n,t_{-n}}$  to  $\mathcal{P}(R_{n,t_n})$ . Since the  $\Omega_t$ 's are closed and disjoint, we must have

$$d_\Omega(\Omega_t, \Omega_{t'}) > 0, \quad \forall t, t' \in T \text{ with } t \neq t'. \quad (19)$$

This allows us to patch up aforementioned functions through all  $t_{-n}$ 's to form  $\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})$  and  $\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})$  as members of  $\Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, \mathcal{P}(R_{n,t_n}))$ . The space  $\mathcal{P}(R_{n,t_n})$  is bounded since the Prokhorov metric  $\psi_{R_{n,t_n}}$  is always below 1. Now for  $\Pi_{n,t_n}$ , we can define the uniform metric, namely, the supremum of all component-wise Prokhorov metrics on  $\mathcal{P}(R_{n,t_n})$ . Under the metric,  $\Pi_{n,t_n}$  is a closed subset of  $(\mathcal{P}(R_{n,t_n}))^{\Omega_{n,t_n}}$ ; see Theorem 43.6 of Munkres [37].

Also, uniform continuities stated in the above will lead to the following.

**Proposition 2** *For any player  $n \in N$  and any of his types  $t_n \in T_n$ , the vector-valued functions  $\pi_{n,t_n}^a$  from  $A_{n,t_n} \times \Delta_{-n}$  to  $\Pi_{n,t_n}$  defined through (9) and  $\pi_{n,t_n}^d$  from  $\Delta_{n,t_n} \times \Delta_{-n}$  to  $\Pi_{n,t_n}$  defined through (13) are both continuous.*

In other words, Proposition 2 propounds two points. First, not every map  $\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})$  or  $\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})$  from states to payoff distributions will arise during any play of the preference game  $\Gamma$ ; rather, only those continuous ones will do. Second, the continuous mappings themselves will react continuously to changes, in own actions  $a_{n,t_n}$  and opponent behavioral-strategy profiles  $\delta_{-n}$  in the action-based case and in own action distributions  $\delta_{n,t_n}$  and opponent behavioral-strategy profiles  $\delta_{-n}$  in the action-based case.

## 4.2 Confined Definition of Preferences

Rather than defined for the entire space  $(\mathcal{P}(R_{n,t_n}))^{\Omega_{n,t_n}}$  of payoff-distribution vectors, we can confine each preference relation  $\succ_{n,t_n}$  to the smaller space  $\Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, \mathcal{P}(R_{n,t_n}))$  of continuous payoff-distribution vectors. For convenience, define the set of pairs

$$G_{n,t_n} = \{(\pi, \pi') \in \Pi_{n,t_n} \times \Pi_{n,t_n} \mid \pi' \not\succ_{n,t_n} \pi\}, \quad (20)$$

where the left members are not less preferential than the right ones. With this definition, an alternative way to express (14) is

$$\hat{A}_{n,t_n}^a = \{a_{n,t_n} \in A_{n,t_n} \mid \{\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})\} \times \tilde{\Pi}_{n,t_n}^a(\delta_{-n}) \subseteq G_{n,t_n}\}, \quad (21)$$

where

$$\tilde{\Pi}_{n,t_n}^a(\delta_{-n}) = [\pi_{n,t_n}^a(\cdot, \delta_{-n})](A_{n,t_n}) \equiv \{\pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}) \mid a'_{n,t_n} \in A_{n,t_n}\}, \quad (22)$$

is the set of potential payoff-distribution vectors in  $\Pi_{n,t_n}$  to be experienced by the  $(n, t_n)$ -player when he tries all possible pure actions in  $A_{n,t_n}$  while his opponents are fixated at the behavioral-strategy profile  $\delta_{-n}$ . Meanwhile, an alternative way to express (17) is

$$\hat{B}_{n,t_n}^d(\delta_{-n}) = \{\delta_{n,t_n} \in \Delta_{n,t_n} \mid \{\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})\} \times \tilde{\Pi}_{n,t_n}^d(\delta_{-n}) \subseteq G_{n,t_n}\}, \quad (23)$$

where

$$\tilde{\Pi}_{n,t_n}^d(\delta_{-n}) = [\pi_{n,t_n}^d(\cdot, \delta_{-n})](\Delta_{n,t_n}) \equiv \{\pi_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}) \mid \delta'_{n,t_n} \in \Delta_{n,t_n}\}, \quad (24)$$

is the set of potential payoff-distribution vectors in  $\Pi_{n,t_n}$  to be experienced by the  $(n, t_n)$ -player when he tries all possible behavioral strategies from  $\Delta_{n,t_n}$  while his opponents are fixated at the behavioral-strategy profile  $\delta_{-n}$ .



Due to  $\pi_{n,t_n}^a(\cdot, \delta_{-n})$ 's continuity as suggested by Proposition 2, the compactness of  $A_{n,t_n}$  will translate into that of  $\tilde{\Pi}_{n,t_n}^a(\delta_{-n})$  defined at (22). Similarly,  $\pi_{n,t_n}^d(\cdot, \delta_{-n})$ 's continuity and  $\Delta_{n,t_n}$ 's compactness will together lead to the compactness of  $\tilde{\Pi}_{n,t_n}^d(\delta_{-n})$  defined at (24). We now make an assumption on the  $\succ_{n,t_n}$ 's as they are defined on the spaces  $\Pi_{n,t_n}$ .

**Preference Assumption 1** *For any player  $n \in N$  and any of his types  $t_n \in T_n$ , the relation  $\succ_{n,t_n}$  is continuous; namely,  $G_{n,t_n}$  defined in (20) is a closed subset of  $\Pi_{n,t_n} \times \Pi_{n,t_n}$ .*

Preference Assumption 1 is routinely treated as part of the definition of a preference. We single it out just to emphasize its importance. The following is an important consequence.

**Proposition 3** *For any player  $n \in N$  and any of his types  $t_n \in T_n$ , a compact  $\Pi' \subseteq \Pi_{n,t_n}$  can always reach  $\succ_{n,t_n}$ -maximal; that is, there exists some  $\pi \in \Pi'$  so that  $\pi' \not\succ_{n,t_n} \pi$  for any  $\pi' \in \Pi'$ , or in other words,  $\{\pi\} \times \Pi' \subseteq G_{n,t_n}$ .*

This result is well known; see e.g., Lemma 2 of Schmeidler [42] and Theorem 5.1 of Khan and Sun [26]. We reproduce it here for the sake of completeness.

### 4.3 Existence Derivations

In view of the compactness of  $\tilde{\Pi}_{n,t_n}^a(\delta_{-n})$ , Proposition 3 will lead to the nonemptiness of  $\hat{A}_{n,t_n}^a(\delta_{-n})$  as defined in (21). Indeed,

$$[\pi_{n,t_n}^a(\cdot, \delta_{-n})](\hat{A}_{n,t_n}^a(\delta_{-n})) \bigcap \tilde{\Pi}_{n,t_n}^a(\delta_{-n}) \neq \emptyset. \quad (25)$$

Note that the nonempty  $\hat{A}_{n,t_n}^a(\delta_{-n})$  is originally defined in (14). So we will have the nonemptiness of  $\hat{B}_{n,t_n}^a(\delta_{-n})$  as well, because by (15), the latter contains the Dirac measure  $1_{a_{n,t_n}}$  for any  $a_{n,t_n} \in \hat{A}_{n,t_n}^a(\delta_{-n})$ . Meanwhile, Preference Assumption 1 will together with the continuity of  $\pi_{n,t_n}^a$  lead to the closedness of  $\hat{A}_{n,t_n}^a$  as a correspondence.

**Proposition 4** *Each  $\hat{A}_{n,t_n}^a(\cdot)$  defined by (14) is closed as a correspondence.*

It turns out that Proposition 4 will lead to the closedness of each correspondence  $\hat{B}_{n,t_n}^a$  as defined in (15), from the space  $\Delta_{-n}$  of opponents' behavioral strategies to the space  $\Delta_{n,t_n}$  of the  $(n, t_n)$ -player's behavioral strategies.

**Proposition 5** *Each  $\hat{B}_{n,t_n}^a$  defined by (15) is closed as a correspondence.*

For the distribution-based case, the definition of  $\hat{B}_{n,t_n}^d(\cdot)$  in (17) is almost the same as that of  $\hat{A}_{n,t_n}^a(\cdot)$  in (14), except with the earlier  $a_{n,t_n}, a'_{n,t_n} \in A_{n,t_n}$  replaced by  $\delta_{n,t_n}, \delta'_{n,t_n} \in \Delta_{n,t_n}$ . So starting from  $\pi_{n,t_n}^d$ 's continuity, we can follow almost the same steps to deduce that each  $\hat{B}_{n,t_n}^d(\delta_{-n})$  is nonempty and that as a correspondence from  $\Delta_{-n}$  to  $\Delta_{n,t_n}$ , each  $\hat{B}_{n,t_n}^d$  is closed. Due to (16) and (18), we can now reach the following conclusion.

**Proposition 6** *The sets  $\hat{B}^a(\delta)$  defined through (16) and  $\hat{B}^d(\delta)$  defined through (18) are both nonempty at any behavioral-strategy profile  $\delta \in \Delta$ ; in addition, both  $\hat{B}^a$  and  $\hat{B}^d$  are closed as correspondences from  $\Delta$  to itself.*

Coming back to the action-based case, the convexity of each  $\hat{B}_{n,t_n}^a(\delta_{-n})$  is obvious from its definition at (15). So via (16) each  $\hat{B}^a(\delta)$  is also convex. On the other hand, each  $\Delta_{n,t_n}$  is a compact and convex subset of the set of signed measures on  $A_{n,t_n}$ , which is itself a locally convex Hausdorff topological vector space. Thus,  $\Delta \equiv \prod_{n \in N} \Delta_n \equiv \prod_{n \in N} \prod_{t_n \in T_n} \Delta_{n,t_n}$  is also a compact and convex subset of a locally convex Hausdorff topological vector space. This makes the closedness of  $\hat{B}^a$  in Proposition 6 equivalent to upper hemi-continuity. Therefore, we can use the Fan-Glicksberg theorem to verify the existence of a fixed point for  $\hat{B}^a$ .

Aside from convexity, the distribution-based case has almost all the properties enjoyed by the action-based case as shown above. To move further, we consider  $\succ_{n,t_n}$  convex when

$$\text{both } \pi \not\succ_{n,t_n} \pi^0 \text{ and } \pi \not\succ_{n,t_n} \pi^1 \text{ will ensure } \pi \not\succ_{n,t_n} (1-\alpha) \cdot \pi^0 + \alpha \cdot \pi^1 \text{ for any } \alpha \in [0, 1]. \quad (26)$$

When  $\succ_{n,t_n}$  is complete, this concept is just ambiguity aversion seen in literature; see Schmeidler [43]. Now by (11) to (13), we have the linearity of  $\pi_{n,t_n}^d(\cdot, \delta_{-n})$ , that

$$\pi_{n,t_n}^d[(1-\alpha) \cdot \delta_{n,t_n}^0 + \alpha \cdot \delta_{n,t_n}^1, \delta_{-n}] = (1-\alpha) \cdot \pi_{n,t_n}^d(\delta_{n,t_n}^0, \delta_{-n}) + \alpha \cdot \pi_{n,t_n}^d(\delta_{n,t_n}^1, \delta_{-n}). \quad (27)$$

In view of (17),  $\hat{B}_{n,t_n}^d(\delta_{-n})$  will be a convex subset of  $\Delta_{n,t_n}$  when  $\succ_{n,t_n}$  is convex. Taking similar steps to those for the action-based case, we can reach the existence of fixed points for  $\hat{B}^d$  as defined at (18). Summing up all of these, we can reach the following conclusion.

**Theorem 1** *The game  $\Gamma$  has action-based equilibria; that is,  $\mathcal{E}^a \neq \emptyset$ . When  $\succ_{n,t_n}$  for every player  $n \in N$  and any of his types  $t_n \in T_n$  is a convex preference relation on  $\Pi_{n,t_n}$ , the game will have distribution-based equilibria, so that  $\mathcal{E}^d \neq \emptyset$ .*

The extra condition needed by Theorem 1 somehow reflects on the “rarity” of distribution-based equilibria in comparison to their action-based brethren.

## 5 Special Cases

### 5.1 The Satisfaction Version

When the preference relations  $\succ_{n,t_n}$  are negatively transitive so that  $\pi \not\succ_{n,t_n} \pi'$  and  $\pi' \not\succ_{n,t_n} \pi''$  always lead to  $\pi \not\succ_{n,t_n} \pi''$ , the relations  $\succ_{n,t_n}$  will become complete pre-orderings. That is, they will be reflexive, transitive, and complete, the latter of which in the sense that either  $\pi \succ_{n,t_n} \pi'$  or  $\pi' \succ_{n,t_n} \pi$  for any  $\pi, \pi' \in \Pi_{n,t_n}$ . Then, it will not take much for order-preserving utility functions over payoff distributions to emerge; see, e.g., Debreu [11].

We suppose such is the case, so that each preference relation  $\succ_{n,t_n}$  is facilitated by a function  $s_{n,t_n}$  through (1) for any continuous payoff-distribution vectors  $\pi, \pi' \in \Pi_{n,t_n}$ . We call these  $s_{n,t_n}$ 's satisfaction functions; also, let us call the game  $\Gamma$ , with the  $s_{n,t_n}$ 's substantiating the  $\succ_{n,t_n}$ 's, a satisfaction game. Define

$$s_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) \equiv s_{n,t_n}(\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})), \quad (28)$$

and

$$s_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}) \equiv s_{n,t_n}(\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})). \quad (29)$$

According to (1), as well as (14) to (16),  $\delta \in \Delta$  will be considered an action-based equilibrium of the satisfaction game if and only if for any player  $n \in N$  and type  $t_n \in T_n$ ,

$$\delta_{n,t_n}(\{a_{n,t_n} \in A_{n,t_n} | s_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) \geq s_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}) \ \forall a'_{n,t_n} \in A_{n,t_n}\}) = 1. \quad (30)$$

By (1), (17), and (18), it will be considered a distribution-based equilibrium of the game if and only if for any player  $n \in N$  and type  $t_n \in T_n$ ,

$$s_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}) \geq s_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}), \quad \forall \delta'_{n,t_n} \in \Delta_{n,t_n}. \quad (31)$$

According to (30), an action-based equilibrium  $\delta^a \in \mathcal{E}^a \subseteq \Delta$  for the satisfaction game  $\Gamma$  is one that, at any  $n \in N$  and  $t_n \in T_n$ , the action distribution  $\delta_{n,t_n}^a$  has its support built on those actions  $a_{n,t_n}^a$  that achieve  $\max_{a_{n,t_n} \in A_{n,t_n}} s_{n,t_n}^a(a_{n,t_n}, \delta_{-n}^a)$ . Meanwhile, according to (31), a distribution-based equilibrium  $\delta^d \in \mathcal{E}^d \subseteq \Delta$  for the game is one that, at any  $n \in N$  and  $t_n \in T_n$ , the action distribution  $\delta_{n,t_n}^d$  achieves  $\max_{\delta_{n,t_n} \in \Delta_{n,t_n}} s_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}^d)$ . For equilibrium existence, let us make a continuity-related assumption.

**Satisfaction Assumption 1** *For any player  $n \in N$  and any of his types  $t_n \in T_n$ , the satisfaction function  $s_{n,t_n}$  is continuous from  $\Pi_{n,t_n}$  to the real line  $\mathbb{R}$ .*

This assumption will imply Preference Assumption 1. So for a satisfaction game, existence of action-based equilibria is guaranteed by Theorem 1. Also, with (1), the  $\succ_{n,t_n}$ -convexity requirement (26) will be equivalent to

$$\begin{aligned} &\text{when both } s_{n,t_n}(\pi) \leq s_{n,t_n}(\pi^0) \text{ and } s_{n,t_n}(\pi) \leq s_{n,t_n}(\pi^1), \\ &\text{it will follow that } s_{n,t_n}(\pi) \leq s_{n,t_n}((1-\alpha) \cdot \pi^0 + \alpha \cdot \pi^1) \text{ for any } \alpha \in [0, 1]. \end{aligned} \quad (32)$$

But this exactly expresses the quasi-concavity of  $s_{n,t_n}$ . Thus, the existence of distribution-based will be guaranteed by Theorem 1 as well under quasi-concave  $s_{n,t_n}$ 's.

**Corollary 1** *The satisfaction game  $\Gamma$  has action-based equilibria. When the satisfaction functions  $s_{n,t_n}$  are quasi-concave, the game will have distribution-based equilibria as well.*

The quasi-concavity of the satisfaction function  $s_{n,t_n}$  translates into the added benefit bestowed to the  $(n, t_n)$ -player when two payoff-distribution vectors are mixed up. Thus, Corollary 1 basically stipulates the nonemptiness of  $\mathcal{E}^a$  for a general satisfaction game and that of  $\mathcal{E}^d$  under ambiguity aversion. The latter message can be used to explain Azrieli and Teper's [5] conclusions for their setting. Since  $\int u_{n,t_n} \cdot d\pi \equiv (\int u_{n,t_n} \cdot d[\pi(\omega)] | \omega \in \Omega_{n,t_n})$  is linear in  $\pi$ , the quasi-concavity of  $j_{n,t_n}$  used in (5) will lead to that of its corresponding  $s_{n,t_n}$ .

## 5.2 The Alarmists' and Enterprising Versions

Suppose at each player  $n \in N$  and type  $t_n \in T_n$ , there exists a real-valued utility function  $u_{n,t_n} \in \mathcal{C}(R_{n,t_n}, \mathbb{R})$  over payoffs and a closed prior set  $P_{n,t_n} \subseteq \mathcal{P}(\Omega_{n,t_n})$  of potential state distributions, so that for any payoff-distribution vector  $\pi \in \Pi_{n,t_n}$ , (3) applies with

$$s_{n,t_n}^0(\pi, \rho) = \int_{\Omega_{n,t_n}} \left\{ \int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\omega)](dr) \right\} \cdot \rho(d\omega). \quad (33)$$

The above integration can be understood as an expectation of the utility  $u_{n,t_n}(R)$  over the random payoff  $R$ , where the latter is distributed according to  $\int_{\Omega_{n,t_n}} \pi(\omega) \cdot \rho(d\omega)$ , essentially components  $\pi(\omega)$  of the vector  $\pi$  mixed over with weights assigned by the state distribution  $\rho$ . We will call this even more special  $\Gamma$  an alarmists' game, because players are effectively on the highest alert to guard against the worst scenario. Under the convexity of  $P_{n,t_n}$  and other technical restrictions, (3) and (33) were demonstrated by Gilboa and Schmeidler [21] to reflect features like certainty independence, continuity, monotonicity, ambiguity aversion, and non-degeneracy to be possessed by the underlying preference relation  $\succ_{n,t_n}$ .

The utility function  $u_{n,t_n} \in \mathcal{C}(R_{n,t_n}, \mathbb{R})$  is not only continuous but also bounded due to the compactness of  $R_{n,t_n}$ . This along with the continuity of  $\pi(\cdot) \in \Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, \mathcal{P}(R_{n,t_n}))$

and the nature of the Prokhorov metric will ensure the continuity of  $\int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\cdot)](dr)$  as a function from  $\Omega_{n,t_n}$  to  $\mathfrak{R}$ :

$$\lim_{\omega' \rightarrow \omega} \int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\omega')](dr) = \int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\omega)](dr), \quad (34)$$

just because  $\lim_{\omega' \rightarrow \omega} \pi(\omega') = \pi(\omega)$ . So the outer-layer integration in (33) is well defined.

Similarly,  $\int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\omega)](dr)$  is continuous in  $\pi(\omega)$  at every  $\omega \in \Omega_{n,t_n}$ . It is also bounded across all these  $\omega$ 's. Moreover, the uniform metric adopted for  $\Pi_{n,t_n}$  means that  $\lim_{k \rightarrow +\infty} \pi^k = \pi$  always entails  $\lim_{k \rightarrow +\infty} \pi^k(\omega) = \pi(\omega)$  at every  $\omega \in \Omega_{n,t_n}$ . With bounded convergence, we will then have  $s_{n,t_n}^0(\cdot, \rho)$ 's continuity as a real-valued function on  $\Pi_{n,t_n}$ . Since the feasible region  $P_{n,t_n}$  in the optimization problem involved in (3) is independent of  $\pi$ , we can immediately deduce the continuity of  $s_{n,t_n}$ . In addition,  $s_{n,t_n}^0(\cdot, \rho)$  is linear in the sense of being both concave and convex. Hence, after taking infimum,  $s_{n,t_n}$  will be concave. The following then comes immediately from Corollary 1.

**Corollary 2** *The alarmists' game  $\Gamma$  has both action-based and distribution-based equilibria.*

With their notation  $\tau_i^{-1}(t_i)$  matching our  $\Omega_{n,t_n}$  and their Equation (4) expressing our (3) here, Kajii and Ui [24] can be understood as studying the alarmists' game with finite action and state spaces. They also listed different ways in which prior sets  $P_{n,t_n} \subseteq \mathcal{P}(\Omega_{n,t_n})$  used in (3) could be generated from prior sets  $P_n \subseteq \mathcal{P}(\Omega)$  defined for the entire state space; e.g., through the fashion of Dempster [12] or the fashion of Fagin and Halpern [19].

Oppositely, we can consider what we shall call the enterprising game, where (4) applies. If the alarmists' game reflects aversion to ambiguity on the actual distribution of states of the world  $\omega$  within the  $\Omega_{n,t_n}$ 's, the enterprising game reflect players' staunch beliefs in the tendency for ambiguities to be resolved in a manner most favorable to them. In other words, the players are really "enterprising". The current  $s_{n,t_n}$ , being linked to  $s_{n,t_n}^0$  through (4), is continuous. The following thus applies.

**Corollary 3** *The enterprising game  $\Gamma$  has action-based equilibria.*

In Sections 7, we will have more to say about this game's pure equilibria, in both action- and distribution-based senses.

### 5.3 The Traditional Expected-utility Version

A special game, which is simultaneously alarmists' and enterprising, is when the prior sets  $P_{n,t_n}$  happen to be singletons  $\{\rho_{n,t_n}\}$ . Then  $\Gamma$ 's satisfaction functions will obey

$$s_{n,t_n}(\pi) = s_{n,t_n}^0(\pi, \rho_{n,t_n}), \quad \forall \pi \in \Pi_{n,t_n}, \quad (35)$$

with  $s_{n,t_n}^0$  given in (33). Note this agrees exactly with (2). This special case turns out just to be the incomplete-information game as understood in the traditional sense.

Let us suppose that

$$p_{n,t_n|t_{-n}} \equiv \int_{\Omega_{t_n,t_{-n}}} \rho_{n,t_n}(d\omega) > 0, \quad \forall n \in N, t \equiv (t_n, t_{-n}) \in T. \quad (36)$$

Since  $\Omega_{n,t_n} = \bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n,t_{-n}}$ ,

$$\sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}} = \sum_{t_{-n} \in T_{-n}} \int_{\Omega_{t_n,t_{-n}}} \rho_{n,t_n}(d\omega) = \int_{\Omega_{n,t_n}} \rho_{n,t_n}(d\omega) = 1, \quad (37)$$

with the last equality attributable to  $\rho_{n,t_n} \in \mathcal{P}(\Omega_{n,t_n})$ . So  $p_{n,t_n} \equiv (p_{n,t_n|t_{-n}})_{t_{-n} \in T_{-n}}$  describes a distribution on opponents' type profiles, wherein every component  $p_{n,t_n|t_{-n}}$  is interpretable as player  $n$ 's estimate on the chance for  $t_{-n}$  to occur at his own type  $t_n$ . Also, let

$$v_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}}) \equiv \int_{\Omega_{t_n,t_{-n}}} u_{n,t_n}(r_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}}, \omega)) \cdot \nu_{n,t_n,t_{-n}}(d\omega), \quad (38)$$

where

$$\nu_{n,t_n,t_{-n}} \equiv \frac{1}{p_{n,t_n|t_{-n}}} \cdot \rho_{n,t_n}|_{\Omega_{t_n,t_{-n}}} \in \mathcal{P}(\Omega_{t_n,t_{-n}}). \quad (39)$$

Here,  $\rho_{n,t_n}|_{\Omega_{t_n,t_{-n}}}$  just means the measure  $\rho_{n,t_n}$  on  $\Omega_{n,t_n}$  being confined to the subset  $\Omega_{t_n,t_{-n}}$ . We can treat each term  $v_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}})$  as player  $n$ 's von Neumann-Morgenstern utility when his own type is  $t_n$ , his opponents' type profile is  $t_{-n}$ , he takes action  $a_{n,t_n}$ , and his opponents collectively adopt action profile  $a_{-n,t_{-n}}$ .

By plugging (6) to (9) into (33), we see that  $s_{n,t_n}^0(\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}), \rho_{n,t_n})$  equals

$$\begin{aligned} & \sum_{t_{-n} \in T_{-n}} \int_{\Omega_{t_n,t_{-n}}} \left\{ \int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega)](dr) \right\} \cdot \rho_{n,t_n}(d\omega) \\ &= \sum_{t_{-n} \in T_{-n}} \int_{\Omega_{t_n,t_{-n}}} \rho_{n,t_n}(d\omega) \times \\ & \quad \times \left\{ \int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [(\prod_{m \neq n} \delta_{m,t_m}) \cdot (r_{n,t_n,t_{-n}}(a_{n,t_n}, \cdot, \omega))^{-1}](dr) \right\}, \end{aligned} \quad (40)$$

which, after a change of variables, an exchange of integration orders, and the use of entities defined in (36) through (39), would become

$$\sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}} \cdot \int_{A_{-n,t_{-n}}} v_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}}) \cdot \prod_{m \neq n} \delta_{m,t_m}(da_{m,t_m}). \quad (41)$$

Similarly, by plugging (10) to (13) into (33) and (41), we can get

$$s_{n,t_n}^0(\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}), \rho_{n,t_n}) = \int_{A_{n,t_n}} s_{n,t_n}^0(\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}), \rho_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n}). \quad (42)$$

From (40) to (42), we see that among  $\Gamma$ 's primitives listed at the end of Section 3.1,  $\Omega$ ,  $(\Omega_{n,t_n})_{n \in N, t_n \in T_n}$ ,  $(R_{n,t_n})_{n \in N, t_n \in T_n}$ ,  $(r_{n,t})_{n \in N, t \in T}$ , and  $(\succ_{n,t_n})_{n \in N, t_n \in T_n}$  are all not needed. In their stead, we can add the probabilities  $p_{n,t_n|t-n}$  and the real-valued payoffs  $v_{n,t_n,t-n}(a_{n,t_n}, a_{-n,t-n})$ 's. These descriptions fit the definition of a traditional game with incomplete information, albeit one without necessarily involving a common prior. Since the current one is a special alarmists' game, we know from Corollary 2 that both action-based and distribution-based equilibria are in existence. Later, we shall see that the linearity of  $s_{n,t_n}^0(\cdot, \rho_{n,t_n})$  will lead the two types of equilibria to exactly overlap.

## 6 Relations between Equilibrium Versions

### 6.1 Useful Definitions

We now come to relationships between  $\mathcal{E}^a$  and  $\mathcal{E}^d$ . Some of the conditions concerning the general preference game might be difficult to check. However, they lead to a clear message about the satisfaction game and ultimately, to the identity of the two equilibrium sets for the traditional game.

Let us first make some definitions. Given metric spaces  $X$ ,  $Y$ , and  $Z$ , let  $\mathcal{K}(X, Y, Z)$  be the space of continuous kernels, so that every  $\kappa \equiv (\kappa(x)|x \in X) \equiv (\kappa(x, y)|x \in X, y \in Y) \in \mathcal{K}(X, Y, Z)$  is a mapping from  $X \times Y$  to  $\mathcal{P}(Z)$  that also satisfies the following:

- (a)  $[\kappa(\cdot, y)](Z')$  is Borel-measurable from  $X$  to  $[0, 1]$  at every  $y \in Y$  and  $Z' \in \mathcal{B}(Z)$ ;
- (b)  $\kappa$  is continuous from  $X \times Y$  to  $\mathcal{P}(Z)$ .

Given  $\delta \in \mathcal{P}(X)$ , we define the integration  $\iota = \int_X \kappa(x) \cdot \delta(dx)$  in the component-wise fashion, so that  $\iota \equiv (\iota(y)|y \in Y)$  and  $\iota(y) = \int_X \kappa(x, y) \cdot \delta(dx)$  at every  $y \in Y$ ; each of the latter integrations, in turn, is facilitated by

$$[\iota(y)](Z') = \int_X [\kappa(x, y)](Z') \cdot \delta(dx), \quad \forall Z' \in \mathcal{B}(Z). \quad (43)$$

Due to (a), the above integration can be carried out. Using (a) and part of (b), we can establish  $\iota$ 's membership in  $\mathcal{C}(Y, \mathcal{P}(Z))$ .

**Lemma 1** *Given  $\kappa \in \mathcal{K}(X, Y, Z)$  and  $\delta \in \mathcal{P}(X)$ , their integration  $\iota$ , whose definition ultimately relies on (43), is a member of  $\mathcal{C}(Y, \mathcal{P}(Z))$ .*

At each  $(n, t_n)$ -pair, it will help to cast the action space  $A_{n,t_n}$  as  $X$ , state space  $\Omega_{n,t_n}$  as  $Y$ , and payoff space  $R_{n,t_n}$  as  $Z$ . We now show that payoffs attained in the action-based sense

are linked to continuous kernels. Given  $\delta_{-n} \in \Delta_{-n}$ , with each  $\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n})$  being patched up in the manner of (9), we can understand the vector  $\pi_{n,t_n}^a(\delta_{-n}) \equiv (\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) | a_{n,t_n} \in A_{n,t_n}) \equiv (\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega) | a_{n,t_n} \in A_{n,t_n}, t_{-n} \in T_{-n}, \omega \in \Omega_{t_n,t_{-n}})$  as a mapping from  $A_{n,t_n} \times \Omega_{n,t_n}$  to  $\mathcal{P}(R_{n,t_n})$ . It turns out to be a continuous kernel.

**Proposition 7** *At any player  $n \in N$ , any of his types  $t_n \in T_n$ , and any of his opponents' behavioral-strategy profiles  $\delta_{-n} \in \Delta_{-n}$ , we have  $\pi_{n,t_n}^a(\delta_{-n}) \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$ .*

Comparing (12) with (43), we can have another understanding of the vector  $\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n})$  defined earlier at (13). Using the current integration of continuous kernels,

$$\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}) = \int_{A_{n,t_n}} \pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) \cdot \delta_{n,t_n}(da_{n,t_n}). \quad (44)$$

Lemma 1 would also predict the vector's membership in  $\Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, R_{n,t_n})$ .

For any preference relation  $\succ_{n,t_n}$  on  $\Pi_{n,t_n}$ , let  $\mathcal{W}_{n,t_n}^a(\succ_{n,t_n})$  be the set that contains all the  $(\kappa_{n,t_n}, \delta_{n,t_n})$ -pairs so that  $\kappa_{n,t_n} \equiv (\kappa_{n,t_n}(a_{n,t_n}) | a_{n,t_n} \in A_{n,t_n})$  is a continuous kernel in  $\mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$ ,  $\delta_{n,t_n}$  is an action distribution in  $\Delta_{n,t_n}$ , and the two satisfy

$$\delta_{n,t_n}(\{a_{n,t_n} \in A_{n,t_n} | \kappa_{n,t_n}(a'_{n,t_n}) \not\succ_{n,t_n} \kappa_{n,t_n}(a_{n,t_n}) \forall a'_{n,t_n} \in A_{n,t_n}\}) = 1, \quad (45)$$

where the set being assessed by  $\delta_{n,t_n}$  is necessarily a member of  $\mathcal{B}(A_{n,t_n})$ . Basically, any  $(\kappa_{n,t_n}, \delta_{n,t_n}) \in \mathcal{W}_{n,t_n}^a(\succ_{n,t_n})$  is such that the payoff-distribution vector  $\kappa_{n,t_n}(a_{n,t_n})$  achieves  $\succ_{n,t_n}$ -maximality among all  $\kappa_{n,t_n}(a'_{n,t_n})$ 's for  $\delta_{n,t_n}$ -almost every  $a_{n,t_n}$ .

Also, let  $\mathcal{W}_{n,t_n}^d(\succ_{n,t_n})$  be the set that contains all the  $(\kappa_{n,t_n}, \delta_{n,t_n})$ -pairs so that  $\kappa_{n,t_n}$  is a continuous kernel in  $\mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$ ,  $\delta_{n,t_n}$  is an action distribution in  $\Delta_{n,t_n}$ , and

$$\int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta'_{n,t_n}(da_{n,t_n}) \not\succ_{n,t_n} \int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n}) \quad \forall \delta'_{n,t_n} \in \Delta_{n,t_n}. \quad (46)$$

By Lemma 1, the integrals are members of  $\Pi_{n,t_n}$ . Basically, any  $(\kappa_{n,t_n}, \delta_{n,t_n}) \in \mathcal{W}_{n,t_n}^d(\succ_{n,t_n})$  is such that the integrated payoff-distribution vector  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n})$  achieves  $\succ_{n,t_n}$ -maximality among all the  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta'_{n,t_n}(da_{n,t_n})$ 's when  $\delta'_{n,t_n}$  traverses through the entire action-distribution space  $\Delta_{n,t_n}$ .

Comparing (14) and (15) with (45), we can understand  $\delta_{n,t_n} \in \hat{B}_{n,t_n}^a(\delta_{-n})$  as

$$(\pi_{n,t_n}^a(\delta_{-n}), \delta_{n,t_n}) \in \mathcal{W}_{n,t_n}^a(\succ_{n,t_n}). \quad (47)$$

Comparing (17) with (46) while observing (44), we can understand  $\delta_{n,t_n} \in \hat{B}_{n,t_n}^d(\delta_{-n})$  as

$$(\pi_{n,t_n}^a(\delta_{-n}), \delta_{n,t_n}) \in \mathcal{W}_{n,t_n}^d(\succ_{n,t_n}). \quad (48)$$



## 6.2 Behavioral Equilibria in General

When  $\mathcal{W}_{n,t_n}^d(\succ_{n,t_n}) \subseteq \mathcal{W}_{n,t_n}^a(\succ_{n,t_n})$ , we say that preference relation  $\succ_{n,t_n}$  is individually prominent with respect to  $A_{n,t_n}$ . As can be seen from (45) and (46), this property entails that  $\kappa_{n,t_n}(a_{n,t_n})$  being not  $\succ_{n,t_n}$ -maximal among all the  $\kappa_{n,t_n}(a'_{n,t_n})$  for a  $\delta_{n,t_n}$ -positive set of  $a_{n,t_n}$ 's would lead to some action distribution  $\delta'_{n,t_n}$  for  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a) \cdot \delta'_{n,t_n}(da)$  to be strictly more preferable than  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a) \cdot \delta_{n,t_n}(da)$ . It will result in the following.

**Proposition 8** *For the preference game  $\Gamma$ , we will have  $\mathcal{E}^d \subseteq \mathcal{E}^a$  when every preference relation  $\succ_{n,t_n}$  is individually prominent with respect to  $A_{n,t_n}$ .*

Oppositely, when  $\mathcal{W}_{n,t_n}^a(\succ_{n,t_n}) \subseteq \mathcal{W}_{n,t_n}^d(\succ_{n,t_n})$ , we say that preference relation  $\succ_{n,t_n}$  is mixture-preserving with respect to  $A_{n,t_n}$ . By (45) and (46), this is when  $\kappa_{n,t_n}(a_{n,t_n})$  being  $\succ_{n,t_n}$ -maximal among all the  $\kappa_{n,t_n}(a'_{n,t_n})$ 's for  $\delta_{n,t_n}$ -almost every  $a_{n,t_n}$  would lead to  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a) \cdot \delta_{n,t_n}(da)$  being  $\succ_{n,t_n}$ -maximal among all the  $\int_{A_{n,t_n}} \kappa_{n,t_n}(a) \cdot \delta'_{n,t_n}(da)$ 's. The property plays a decisive role in the other direction of equilibrium-set inclusion.

**Proposition 9** *For the preference game  $\Gamma$ , we will have  $\mathcal{E}^a \subseteq \mathcal{E}^d$  when every preference relation  $\succ_{n,t_n}$  is mixture-preserving with respect to  $A_{n,t_n}$ .*

When the relations  $\succ_{n,t_n}$  are facilitated by satisfaction functions  $s_{n,t_n}$ , we will show that individual prominence is linked to ambiguity seeking. Meanwhile, mixture preservation seems more stringent as so far its guarantors involve both ambiguity aversion and seeking. In this backdrop, Propositions 8 and 9 may be found to be consistent with the notion that distribution-based equilibria are “rarer” than their action-based counterparts.

Due to (b), any continuous kernel  $\kappa_{n,t_n} \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$  can be viewed as a continuous mapping from  $A_{n,t_n}$  to  $\Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, \mathcal{P}(R_{n,t_n}))$ . Now consider satisfaction function  $s_{n,t_n}$  defined on  $\Pi_{n,t_n}$  that meets Satisfaction Assumption 1, i.e., the continuity of  $s_{n,t_n}$  as a function from  $\Pi_{n,t_n}$  to  $\mathbb{R}$ . Then,  $s_{n,t_n}(\kappa_{n,t_n}(\cdot))$  is a continuous and hence measurable mapping from  $A_{n,t_n}$  to  $\mathbb{R}$ . We say such  $s_{n,t_n}$  strongly concave with respect to  $A_{n,t_n}$  when for any continuous kernel  $\kappa_{n,t_n} \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$  and action distribution  $\delta_{n,t_n} \in \Delta_{n,t_n}$ ,

$$s_{n,t_n} \left( \int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n}) \right) \geq \int_{A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) \cdot \delta_{n,t_n}(da_{n,t_n}). \quad (49)$$

We say  $s_{n,t_n}$  strongly convex with respect to  $A_{n,t_n}$  when the inequality opposite to (49) is always true. We say  $s_{n,t_n}$  strongly linear with respect to  $A_{n,t_n}$  when it is both strongly concave and convex with respect to  $A_{n,t_n}$ . Here come the links between  $s_{n,t_n}$ 's strong properties and earlier notions about the  $s_{n,t_n}$ -based preference relation  $\succ_{n,t_n}$ .

**Proposition 10** *With respect to the same  $A_{n,t_n}$ , any preference relation  $\succ_{n,t_n}$  for  $\Pi_{n,t_n}$  that is based on a strongly convex satisfaction function  $s_{n,t_n}$  will be individually prominent.*

**Proposition 11** *With respect to the same  $A_{n,t_n}$ , any preference relation  $\succ_{n,t_n}$  for  $\Pi_{n,t_n}$  that is based on a strongly linear satisfaction function  $s_{n,t_n}$  will be both individually prominent and mixture-preserving.*

So far, we have not found any intermediate result which guarantees mixture preservation without ensuring individual prominence. On the other hand, strong concavity/convexity of  $s_{n,t_n}$  with respect to  $A_{n,t_n}$  is certainly stronger than  $s_{n,t_n}$ 's ordinary concavity/convexity except when  $A_{n,t_n}$  is a singleton, at which time the strong properties reduce to truisms. It turns out that the converse is actually true.

**Proposition 12** *For any player  $n \in N$  and any of his types  $t_n \in T_n$ . Suppose  $s_{n,t_n}$  is a satisfaction function over  $\Pi_{n,t_n}$  that meets Satisfaction Assumption 1. Then, its ordinary concavity/convexity will lead to its strong concavity/convexity.*

Combining Propositions 8, 10, and 12, we see that  $\mathcal{E}^d \subseteq \mathcal{E}^a$  will happen when the satisfaction functions are convex; whereas, combining everything from Propositions 8, 9, 11, and 12, we see that both the previous and  $\mathcal{E}^a \subseteq \mathcal{E}^d$  will happen when satisfaction functions are linear. We can now reach the following for the satisfaction game introduced in Section 5.1.

**Theorem 2** *For the satisfaction game  $\Gamma$ , we have  $\mathcal{E}^d \subseteq \mathcal{E}^a \neq \emptyset$  when the satisfaction functions  $s_{n,t_n}$  are convex; furthermore, we have  $\mathcal{E}^d = \mathcal{E}^a \neq \emptyset$  when the functions are linear.*

Note that  $\mathcal{E}^a \neq \emptyset$  is attributable to Corollary 1. Now the message is clear for the satisfaction game. Any distribution-based equilibrium will be an action-based one when all players are ambiguity-seeking. When viewing Theorem 2 in conjunction with Corollary 1, we may speculate that distribution-based equilibria are “easier” to come by when players are “more” ambiguity-averse. The theorem also leads to more understanding on the enterprising game studied in Section 5.2.

**Corollary 4** *For the enterprising game  $\Gamma$ , we have  $\mathcal{E}^d \subseteq \mathcal{E}^a \neq \emptyset$ .*

Although the nonemptiness of  $\mathcal{E}^a$  is guaranteed, so far we can not claim the same for the enterprising game's  $\mathcal{E}^d$ . On the other hand, as shall be clear in Section 7, ambiguity seeking will not prevent distribution-based equilibria from emerging. When players are ambiguous-neutral, Theorem 2 also portends the convergence of the two equilibrium concepts.

**Corollary 5** *For the traditional game  $\Gamma$ , we have  $\mathcal{E}^d = \mathcal{E}^a \neq \emptyset$ .*

The nonemptiness of both equilibrium types comes again from Theorem 1. Corollary 5 offers the justification on why traditionally, one does not have worry too much about how behavioral equilibria are interpreted and enforced.

### 6.3 Pure Equilibria in Particular

When focusing on pure equilibria where players do not use chance outcomes in their own strategies, we have a simple conclusion. When action distribution  $\delta_{n,t_n} \in \Delta_{n,t_n}$  happens to be the Dirac measure  $1_{a_{n,t_n}}$  concentrating on one pure action  $a_{n,t_n} \in A_{n,t_n}$ , we have from (11) and (12) that

$$\pi_{n,t_n,t_{-n}}^d(1_{a_{n,t_n}}, \delta_{-n,t_{-n}}, \omega) = \pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega). \quad (50)$$

So in view of (9) and (13),

$$\pi_{n,t_n}^d(1_{a_{n,t_n}}, \delta_{-n}) = \pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}). \quad (51)$$

This simple observation would render it necessary that a pure distribution-based equilibrium is also a pure action-based equilibrium. For convenience, use  $1_{a_{-n,t_{-n}}}$  for  $(1_{a_{m,t_m}} | m \neq n)$  and  $1_{a_{-n}}$  for  $(1_{a_{-n,t_{-n}}})_{t_{-n} \in T_{-n}}$ . Also, let  $1_{A_{n,t_n}} \equiv \{1_{a_{n,t_n}} | a_{n,t_n} \in A_{n,t_n}\}$ ,  $1_{A_n} \equiv \prod_{t_n \in T_n} 1_{A_{n,t_n}}$ , and  $1_A \equiv \prod_{n \in N} 1_{A_n}$ . The last is the space of all pure strategy profiles.

**Theorem 3** *For the preference game  $\Gamma$ , we have  $1_A \cap \mathcal{E}^d \subseteq 1_A \cap \mathcal{E}^a$ .*

Let us revisit the enterprising game defined in Section 5.2. Recall the definition of  $s_{n,t_n}^a$  at (28) and that of  $s_{n,t_n}^d$  at (29).

**Proposition 13** *It follows that  $s_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) = s_{n,t_n}^d(1_{a_{n,t_n}}, \delta_{-n})$ .*

When the action spaces  $A_{n,t_n}$  are finite, the behavioral-strategy spaces  $\Delta_{n,t_n}$  are simplices embedded in  $\mathbb{R}^{A_{n,t_n}}$ . Then, pure strategies in  $1_{A_{n,t_n}}$  constitute all extreme points of  $\Delta_{n,t_n}$ . As the supremums of convex functions over convex sets come from extreme points,

$$\sup_{\delta'_{n,t_n} \in 1_{A_{n,t_n}}} s_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}) = \sup_{\delta'_{n,t_n} \in \Delta_{n,t_n}} s_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}). \quad (52)$$

For a more general case, we resort to the following result.

**Lemma 2** *Suppose  $X$  is a compact subset of a finite-dimensional real Euclidean space and  $f$  is a continuous and convex real-valued function defined on  $\mathcal{P}(X)$ . Then,  $\sup_{\xi \in \mathcal{P}(X)} f(\xi)$  can be achieved at the Dirac measure  $1_x$  for some  $x \in X$ .*

In the enterprising game  $\Gamma$ , we now suppose that all action spaces  $A_{n,t_n}$  are compact subsets of finite-dimensional real Euclidean spaces. Note that  $s_{n,t_n}^d(\cdot, \delta_{-n})$  is not only convex but also continuous. For the latter, just follow the continuity of  $\pi_{n,t_n}^d(\cdot, \delta_{-n})$  as stated in Proposition 2,  $s_{n,t_n}^d(\cdot, \delta_{-n})$ 's definition at (29), and the continuity of the  $s_{n,t_n}$  defined at (4) that was covered right before Corollary 3. By identifying the function  $s_{n,t_n}^d(\cdot, \delta_{-n})$  with  $f$  in Lemma 2 and the set  $A_{n,t_n}$  with  $X$  in the lemma, we can again reach (52). This and the earlier Proposition 13 turn out to be pivotal for the opposite of either Corollary 4 or Theorem 3. We can obtain the following when all these are combined.

**Theorem 4** *For the enterprising game  $\Gamma$ , we have  $1_A \cap \mathcal{E}^d = 1_A \cap \mathcal{E}^a$ .*

Therefore, there will be a unified set  $1_{\mathcal{E}} \equiv 1_A \cap \mathcal{E}^d = 1_A \cap \mathcal{E}^a$  of pure equilibria for the enterprising game when action spaces are mildly regulated. For a special case, not only is  $1_{\mathcal{E}}$  nonempty, but we can also identify from it those well-behaving ones.

## 7 A Special Enterprising Game

### 7.1 No Ambiguity on Opponent-type Distributions

Due to Theorem 4's unification of its two types of pure equilibria, we only have to deal with pure action-based equilibria for the enterprising game. Define

$$\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n}) = s_{n,t_n}^a(a_{n,t_n}, 1_{a_{-n}}), \quad (53)$$

where  $s_{n,t_n}^a$  is given by (28) and  $a_{-n} \in \prod_{t_{-n} \in T_{-n}} A_{-n,t_{-n}} \equiv \prod_{m \neq n} \prod_{t_m \in T_m} A_{m,t_m}$  represents opponents' pure-action profile. Due to (4), (6) to (9), (28), (33), and (53),

$$\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n}) = \sup_{\rho \in P_{n,t_n}} w_{n,t_n}(a_{n,t_n}, a_{-n}, \rho), \quad (54)$$

where

$$w_{n,t_n}(a_{n,t_n}, a_{-n}, \rho) = \sum_{t_{-n} \in T_{-n}} \int_{\Omega_{t_n, t_{-n}}} \tilde{u}_{n,t_n, t_{-n}}(a_{n,t_n}, a_{-n, t_{-n}}, \omega) \cdot \rho|_{\Omega_{t_n, t_{-n}}}(d\omega), \quad (55)$$

and  $\tilde{u}_{n,t_n, t_{-n}} \equiv u_{n,t_n} \circ r_{n,t_n, t_{-n}}$  is the continuous real-valued composite payoff-utility function defined on the compact  $A_t \times \Omega_t \equiv A_{n,t_n} \times A_{-n, t_{-n}} \times \Omega_{t_n, t_{-n}}$ .

By (30), we will have pure strategy  $1_a \in 1_A \cap \mathcal{E}^a$  if and only if  $a_{n,t_n} \in \tilde{B}_{n,t_n}(a_{-n})$ , where

$$\tilde{B}_{n,t_n}(a_{-n}) = \{a_{n,t_n} \in A_{n,t_n} | \tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n}) \geq \tilde{s}_{n,t_n}(a'_{n,t_n}, a_{-n}) \ \forall a'_{n,t_n} \in A_{n,t_n}\}. \quad (56)$$

Thus,  $1_a \in 1_A$  will be a pure equilibrium for  $\Gamma$  if and only if  $a \in A$  is that for a corresponding agent-based normal-form game where payoffs are given by the  $\tilde{s}_{n,t_n}$ 's given at (54) and (55).

We find a special case to be further analyzable. In this case,

- (a) all the action spaces  $A_{n,t_n}$ 's across different  $t_n$ 's are the same;
- (b) there is a compact metric space  $\tilde{\Omega}$ , so that every  $\Omega_t$  is merely  $\{t\} \times \tilde{\Omega}$ ;
- (c) for each player  $n \in N$  and type  $t_n \in T_n$ , there are distribution  $p_{n,t_n} \equiv (p_{n,t_n|t_{-n}})_{t_{-n} \in T_{-n}}$  and nonempty subset  $\mathcal{Q}_{n,t_n}$  of  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ , so that the prior set used in the definition (4), as a nonempty subset of  $\mathcal{P}(\Omega_{n,t_n}) \equiv \mathcal{P}(T_{-n} \times \tilde{\Omega})$ , satisfies

$$P_{n,t_n} = \{\text{concatenation of the } p_{n,t_n|t_{-n}} \times 1_{t_{-n}} \times \nu_{t_{-n}} \text{'s} | \nu \equiv (\nu_{t_{-n}})_{t_{-n} \in T_{-n}} \in \mathcal{Q}_{n,t_n}\}. \quad (57)$$

In most works on games involving incomplete information, (a) was assumed. Due to this point, we can just use  $A_n$  for the action space of player  $n$  regardless of his type and use  $A_{-n}$  for  $\prod_{m \neq n} A_m$ . By (b), every  $\Omega_{n,t_n} = T_{-n} \times \tilde{\Omega}$  and  $\Omega = T \times \tilde{\Omega}$ , indicating that clear cuts can be made between players' types and other external factors which affect all players. For convenience, we still call each  $\tilde{\omega}$  a state. The domain of every payoff-utility function  $\tilde{u}_{n,t_n,t_{-n}}$  is  $A_n \times A_{-n} \times \Omega_{n,t_n,t_{-n}}$ . Because different  $\Omega_{n,t_n,t_{-n}}$ 's are disjoint, we can patch up all the  $\tilde{u}_{n,t_n,t_{-n}}$  to obtain  $\tilde{u}_n : A_n \times A_{-n} \times \Omega \rightarrow \mathbb{R}$ . But with (b),  $\omega = (t_n, t_{-n}, \tilde{\omega})$ . So the just gotten  $\tilde{u}_n(a_n, a_{-n}, \omega)$  can be further rewritten as  $\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega})$ . This function is still continuous on a compact space. Meanwhile, (c) means the following. With probability  $p_{n,t_n|t_{-n}}$  player  $n$  believes unambiguously that opponents' type profile is at some  $t_{-n}$ ; his ambiguity on other external factors, on the other hand, is reflected by the membership of the prior vector  $\nu \equiv (\nu_{t_{-n}})_{t_{-n} \in T_{-n}}$  in the set  $\mathcal{Q}_{n,t_n}$ .

With (57) in place, (54) and (55) can be rewritten as

$$\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n}) = \sup_{\nu \in \mathcal{Q}_{n,t_n}} \tilde{w}_{n,t_n}(a_{n,t_n}, a_{-n}, \nu), \quad (58)$$

where  $a_{n,t_n} \in A_n$ ,  $a_{-n} \equiv (a_{-n,t_{-n}})_{t_{-n} \in T_{-n}} \equiv (a_{m,t_m})_{m \neq n, t_m \in T_m} \in (A_{-n})^{T_{-n}} \equiv \prod_{m \neq n} A_m^{T_m}$ , and  $\nu \equiv (\nu_{t_{-n}})_{t_{-n} \in T_{-n}} \in (\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ ; also,

$$\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu) = \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}} \cdot \tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}), \quad (59)$$

where  $a_n \in A_n$  and

$$\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu) = \int_{\tilde{\Omega}} \tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}) \cdot \mu(d\tilde{\omega}), \quad (60)$$

where this time  $a_{-n} \in A_{-n}$  and  $\mu \in \mathcal{P}(\tilde{\Omega})$ .

Later, it might help to understand the  $\mathcal{Q}_{n,t_n}$  used in (57) as follows:

$$\mathcal{Q}_{n,t_n} = \left( \prod_{t_{-n} \in T_{-n}} \tilde{P}_{n,t_n,t_{-n}} \right) \cap \mathcal{K}_{n,t_n}, \quad (61)$$

where  $\tilde{P}_{n,t_n,t_{-n}} \subseteq \mathcal{P}(\tilde{\Omega})$  for each  $t_{-n} \in T_{-n}$  and  $\mathcal{K}_{n,t_n} \subseteq (\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ . For instance, we can always let  $\tilde{P}_{n,t_n,t_{-n}} = \mathcal{P}(\tilde{\Omega})$  and  $\mathcal{K}_{n,t_n} = \mathcal{Q}_{n,t_n}$ . Our special enterprising game is denotable by  $\Gamma \equiv (N, (T_n)_{n \in N}, (A_n)_{n \in N}, \tilde{\Omega}, (\tilde{u}_{n,t})_{n \in N, t \in T}, (p_{n,t_n})_{n \in N, t_n \in T_n}, (\tilde{P}_{n,t})_{n \in N, t \in T}, (\mathcal{K}_{n,t_n})_{n \in N, t_n \in T_n})$ . In it, there is no ambiguity on the opponent-type distribution. Rather, each  $(n, t_n)$ -player is uncertain about distributions of the non-type factors.

## 7.2 Enter Strategic Complementarities

Let each action space  $A_n$  be a finite set or compact interval within the real line  $\mathfrak{R}$ , and equip it with the ordinary order. For each  $A_{-n}$ , we adopt the component-wise partial order. For two partially ordered sets  $X$  and  $Y$ , we use  $\mathcal{M}(X, Y)$  to denote the subset of  $Y^X$  that contains all monotone mappings from  $X$  to  $Y$ , i.e., mappings  $y : X \rightarrow Y$  so that  $y(x^1) \leq y(x^2)$  whenever  $x^1, x^2 \in X$  satisfy  $x^1 \leq x^2$ . We let the component-wise partial order be adopted for  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$  as well.

We further suppose that the state space  $\tilde{\Omega} = \prod_{k=1}^{\bar{k}} \tilde{\Omega}_k$  where  $\bar{k}$  is a natural number and each  $\tilde{\Omega}_k$  is a finite set or compact interval within the real line  $\mathfrak{R}$ . Also, we equip  $\tilde{\Omega}$  with the component-wise partial order. For the state-distribution space  $\mathcal{P}(\tilde{\Omega})$ , we adopt the usual stochastic order, so that  $\mu^1, \mu^2 \in \mathcal{P}(\tilde{\Omega})$  is considered to satisfy  $\mu^1 \leq \mu^2$  when for any monotone function  $q \in \mathcal{M}(\tilde{\Omega}, \mathfrak{R})$  that is integrable under both  $\mu^1$  and  $\mu^2$ ,

$$\int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^1(d\tilde{\omega}) \leq \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^2(d\tilde{\omega}). \quad (62)$$

The above is equivalent to  $\mu^1(\tilde{\Omega} \cap U) \leq \mu^2(\tilde{\Omega} \cap U)$  for every of  $\mathfrak{R}^{\bar{k}}$ 's upper sets  $U$ , a set satisfying  $\omega^2 \in U$  whenever  $\omega^1 \in U$  and  $\omega^1 \leq \omega^2$ ; see, e.g., Section 6.B.1 of Shaked and Shanthikumar [45]. Given  $\mu^1, \mu^2 \in \mathcal{P}(\tilde{\Omega})$ , we can construct  $\mu^1 \vee \mu^2 \in \mathcal{P}(\tilde{\Omega})$  by forcing its value at  $\tilde{\Omega} \cap K$  for every upper rectangular set  $K$  be  $\mu^1(\tilde{\Omega} \cap K) \vee \mu^2(\tilde{\Omega} \cap K)$ . Similarly, we can obtain  $\mu^1 \wedge \mu^2 \in \mathcal{P}(\tilde{\Omega})$ . Thus,  $\mathcal{P}(\tilde{\Omega})$  is a lattice under the usual stochastic order. For a partial order between sublattices of  $\mathcal{P}(\tilde{\Omega})$ , we can adopt the induced set order; see Theorem 2.4.1 of Topkis [48]. For sublattices  $P^1$  and  $P^2$  of  $\mathcal{P}(\tilde{\Omega})$ , we consider  $P^1 \leq P^2$  in the induced set order sense when  $\mu^1 \in P^1$  and  $\mu^2 \in P^2$  will always lead to

$$\mu^1 \wedge \mu^2 \in P^1, \quad \mu^1 \vee \mu^2 \in P^2. \quad (63)$$

We can adopt the component-wise partial order for each lattice  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ . This way,  $\nu^1 \equiv (\nu_{t_{-n}}^1)_{t_{-n} \in T_{-n}}, \nu^2 \equiv (\nu_{t_{-n}}^2)_{t_{-n} \in T_{-n}} \in (\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  are considered to satisfy  $\nu^1 \leq \nu^2$  when

$$\nu_{t_{-n}}^1 \leq \nu_{t_{-n}}^2, \quad \forall t_{-n} \in T_{-n}. \quad (64)$$

This partial order certainly applies to the smaller lattice  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  as well. For the latter's sublattices, we can similarly adopt the induced set order.

We now make further assumptions on the game's model primitives.

**Monotonic Assumption 1** *For any  $n \in N$ ,  $t_n \in T_n$ ,  $t_{-n} \in T_{-n}$ ,  $a_n \in A_n$ , and  $a_{-n} \in A_{-n}$ , the payoff-utility  $\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega})$  is increasing in  $\tilde{\omega} \in \tilde{\Omega}$ .*

**Monotonic Assumption 2** *For any  $n \in N$ , the payoff-utility function  $\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega})$  has increasing differences between  $a_n \in A_n$  and  $(t_n, t_{-n}, a_{-n}, \tilde{\omega}) \in T_n \times T_{-n} \times A_{-n} \times \tilde{\Omega}$ , as well as between  $(t_n, t_{-n}, a_{-n}) \in T_n \times T_{-n} \times A_{-n}$  and  $\tilde{\omega} \in \tilde{\Omega}$ .*

**Monotonic Assumption 3** *For any  $n \in N$ , the distribution  $p_{n,t_n}$  is monotone in  $t_n \in T_n$  in the usual stochastic order, so that for any  $t_n^1, t_n^2 \in T_n$  with  $t_n^1 \leq t_n^2$  and any  $f \in \mathcal{M}(T_{-n}, \mathfrak{R})$ ,*

$$\sum_{t_{-n} \in T_{-n}} p_{n,t_n^1|t_{-n}} \cdot f_{t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n^2|t_{-n}} \cdot f_{t_{-n}}.$$

**Monotonic Assumption 4** *For any  $n \in N$ ,  $t_n \in T_n$ , and  $t_{-n} \in T_{-n}$ , the prior set  $\tilde{P}_{n,t_n,t_{-n}}$  is a sublattice of the lattice  $\mathcal{P}(\tilde{\Omega})$ .*

**Monotonic Assumption 5** *For any  $n \in N$  and  $t_{-n} \in T_{-n}$ , the prior set  $\tilde{P}_{n,t_n,t_{-n}}$  is increasing in  $t_n \in T_n$ .*

Monotonic Assumption 1 essentially associates higher  $\tilde{\omega}$  values with better payoffs. In Monotonic Assumption 2, the payoff-utility function's increasing differences between player  $n$ 's own action  $a_n$  and the type-action profile  $(t_n, t_{-n}, a_{-n})$  is quite anticipated for a game involving strategic complementarities. They indicate the increasing efficiency of a player under ever more friendly environments. These properties are also required in the traditional expected-utility version as well; see, e.g., van Zandt and Vives [52]. The full plate of increasing differences involving the newly added factor  $\tilde{\omega}$ , which resemble those for the action  $a_n$ , suggest that the latter should bear the interpretation of not only an efficiency booster but also somehow a surrogate action.

Note that the action space  $A_n$  is a subset of the single-dimensional real line; also,  $\tilde{u}_{n,t_n,t_{-n}}$  is already assumed to be continuous. So we have no need for additional supermodularity

and continuity requirements on  $\tilde{u}_{n,t_n,t_{-n}}(\cdot, a_{-n}, \tilde{\omega})$ . For any  $\tilde{u}_{n,t_n,t_{-n}}^0(a_n, a_{-n})$  already suitable as a payoff function for the traditional game,  $\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega})$  defined in terms of

$$\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}) = \tilde{u}_{n,t_n,t_{-n}}^0(a_n, a_{-n}) + (\alpha_n t_n + \beta_n t_{-n} + \gamma_n a_n + \nu_n a_{-n} + \zeta_n) \cdot \tilde{\omega}, \quad (65)$$

where  $\alpha_n$ ,  $\beta_n$ ,  $\gamma_n$ , and  $\nu_n$  are positive constants, and  $\zeta_n$  is a constant guaranteeing the positivity of the entire multiplier in front of  $\tilde{\omega}$ , will satisfy Monotonic Assumption 2.

Meanwhile, Monotonic Assumption 3 suggests that a player's own type is positively correlated with his opponents' types. It has been assumed for the traditional game as well; see van Zandt and Vives [52]. Finally, Monotonic Assumptions 4 and 5 collectively indicate that a player's own type is positively correlated with the external factors. Taken together, the latter two points both highlight the informational value of a player's own type.

### 7.3 Monotone Pure Equilibria

Now we can obtain an intermediate result of the order-theoretic nature.

**Proposition 14** *For  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu)$  defined at (60), it is both supermodular and submodular in  $\mu \in \mathcal{P}(\tilde{\Omega})$ . Also, it has increasing differences between  $a_n \in A_n$  and  $(t_n, t_{-n}, a_{-n}, \mu) \in T_n \times T_{-n} \times A_{-n} \times \mathcal{P}(\tilde{\Omega})$ , as well as between  $(t_n, t_{-n}, a_{-n}) \in T_n \times T_{-n} \times A_{-n}$  and  $\mu \in \mathcal{P}(\tilde{\Omega})$ .*

To go any further, however, we find it necessary to consider separately two special scenarios. In scenario A, the distributions  $p_{n,t_n}$  are independent of  $t_n$ . We can thus use  $p_{n|t_{-n}}^A$  to stand for each probability  $p_{n,t_n|t_{-n}}$ . Also, the prior sets  $\tilde{P}_{n,t_n,t_{-n}}$  can be some general  $\tilde{P}_{n,t_n,t_{-n}}^A$ 's. However, each  $\mathcal{K}_{n,t_n}$  is equal to the set of monotone mappings  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ . In view of (61), the latter two facts together lead to

$$\mathcal{Q}_{n,t_n}^A \equiv \left( \prod_{t_{-n} \in T_{-n}} \tilde{P}_{n,t_n,t_{-n}}^A \right) \cap \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega})). \quad (66)$$

Here, player  $n$  should not expect to gain from the identity of his own type  $t_n$  any information about opponents' types  $t_{-n}$ ; yet, he should anticipate the latter types to be positively correlated with the external factor  $\tilde{\omega}$ .

In scenario B, the probabilities  $p_{n,t_n|t_{-n}}$ 's can be some general  $p_{n,t_n|t_{-n}}^B$ 's. However, the prior set  $\tilde{P}_{n,t_n,t_{-n}}$  is independent of  $t_{-n}$ , and hence is representable by  $\tilde{P}_{n,t_n}^B$ . In addition, each  $\mathcal{K}_{n,t_n}$  is equal to  $1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , the set of constant mappings from  $T_{-n}$  to  $\mathcal{P}(\tilde{\Omega})$ . Note that  $1(T_{-n}, \mathcal{P}(\tilde{\Omega})) \subseteq \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ . In view of (61), the above would lead to

$$\mathcal{Q}_{n,t_n}^B \equiv \left( (\tilde{P}_{n,t_n}^B)^{T_{-n}} \right) \cap 1(T_{-n}, \mathcal{P}(\tilde{\Omega})). \quad (67)$$



Here, player  $n$  can learn from his own type  $t_n$  something about opponents' types  $t_{-n}$ ; yet, these latter types will play no role in shaping his understanding of the external factor  $\tilde{\Omega}$ .

**Proposition 15** *For  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  defined at (59), it is both supermodular and submodular in  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , and has increasing differences between  $a_n \in A_n$  and  $(t_n, a_{-n}, \nu) \in T_n \times \prod_{m \neq n} \mathcal{M}(T_m, A_m) \times \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , as well as between  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ . In addition, the function has increasing differences between  $t_n \in T_n$  and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario A and between  $t_n \in T_n$  and  $\nu \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario B.*

**Proposition 16** *For  $\mathcal{Q}_{n,t_n}$  defined at (61), regardless of the scenario that prevails, at each fixed  $t_n$  it is a nonempty sublattice of  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  and hence a nonempty lattice in its own right. Also, it is increasing in  $t_n$ .*

It is noteworthy that in Proposition 15, we have restricted  $a_{-n} \in \prod_{m \neq n} A_m^{T_m}$  to monotone opponent strategies in  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu \in (\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  to monotone opponent-type-to-state-distribution maps in  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ . Moreover, the need there to prove the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  has between  $t_n$  and  $\nu$  has prevented us from considering the more general case, where the conditional probabilities  $p_{n,t_n|t_{-n}}$  take the more general B-version and the prior sets  $\mathcal{Q}_{n,t_n}$  take the more general A-version.

The following lemma is about the preservation of increasing differences after maximization in the nature of (58). It is likely to be useful in other circumstances.

**Lemma 3** *Given partially ordered sets  $X$  and  $Y$ , as well as lattice  $Z$ , let  $f$  be a real-valued function defined on  $X \times Y \times Z$ , and  $\tilde{Z}(y)$  be a subset of  $Z$  at each  $y \in Y$ . Suppose that (I)  $f$  has increasing differences between  $x \in X$  and  $(y, z) \in Y \times Z$ , that (II)  $f$  is supermodular in  $z \in Z$ , that (III) each  $\tilde{Z}(y)$  is a sublattice, and that (IV)  $\tilde{Z}(\cdot)$  is increasing in  $y$ . Also, suppose that (V)  $f$  has increasing differences between  $y \in Y$  and  $z \in Z$ . Then, for*

$$g(x, y) = \sup_{z \in \tilde{Z}(y)} f(x, y, z),$$

*it will follow that  $g$  has increasing differences between  $x \in X$  and  $y \in Y$ .*

We have singled out hypothesis (V) in Lemma 3 regarding  $f$ 's increasing differences between  $y$  and  $z$ , because it seems the most demanding to us. This is the reason why Proposition 15 is concerned even with increasing differences between  $a_{-n}$  and  $\nu$ , which ripple back to similar requirements in Proposition 14 and to Monotonic Assumption 2. Note that Theorem 2.7.6 of Topkis [48] goes from the supermodularity of  $f$  in  $(x, y)$  and lattice nature of  $Y$  to the supermodularity of  $g$  as defined in  $g(x) = \sup_{y \in Y} f(x, y)$ . Our result is of a similar nature. With it, we can obtain a result key to equilibrium analysis.

**Proposition 17** *For  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n})$  defined at (58), it has increasing differences between  $a_n \in A_n$  and  $(t_n, a_{-n}) \in T_n \times \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ .*

Recall that  $\tilde{u}_{n,t_n,t_{-n}}$  is continuous on a compact space. Through (58) to (60), this will lead to the continuity of  $\tilde{s}_{n,t_n}(\cdot, a_{-n})$ . Since each  $A_n$  is a finite set or closed interval within the real line, not only is each  $A_n$  a complete lattice but  $\tilde{s}_{n,t_n}(\cdot, a_{-n})$  is also supermodular. Now the complete-lattice nature of the  $A_n$ 's, as well as the continuity, supermodularity, and Proposition 17's increasing differences will have provided essential elements of a game possessing strategic complementarities. Our ensuing analysis can lean on existing works such as Milgrom and Roberts [33], Milgrom and Shannon [34], and Yang and Qi [51].

Combining Theorems 1 and 2 of Milgrom and Roberts [33] (also summarized as Fact 2 of Yang and Qi [51]), we can conclude that each best-response action set  $\tilde{B}_{n,t_n}(a_{-n})$  defined at (56) is a nonempty complete sublattice of  $A_n$ . By Milgrom and Shannon [34] (also summarized as Fact 3 of Yang and Qi [51]), we further know that  $\tilde{B}_{n,t_n}(a_{-n})$  is increasing in  $(t_n, a_{-n}) \in T_n \times \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ . The remainder of the development closely follows Yang and Qi [51]. As noted by it, each  $\mathcal{M}(T_n, A_n)$ , the space of monotone type-to-action mappings of player  $n$ , is a complete lattice. Now for any  $n \in N$ , define correspondence  $\tilde{\mathcal{B}}_n : \prod_{m \neq n} \mathcal{M}(T_m, A_m) \rightrightarrows \mathcal{M}(T_n, A_n)$  from the space of monotone type-to-action mappings of other players to the space of the current player's monotone type-to-action mappings:

$$\tilde{\mathcal{B}}_n(a_{-n}) = \{a_n \equiv (a_{n,t_n})_{t_n \in T_n} \in \mathcal{M}(T_n, A_n) | a_{n,t_n} \in \tilde{B}_{n,t_n}(a_{-n}) \forall t_n \in T_n\}, \quad (68)$$

for any  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ . The following is a useful characterization.

**Proposition 18** *For the correspondence  $\tilde{\mathcal{B}}_n$  defined at (68), it is a nonempty complete sublattice of  $\mathcal{M}(T_n, A_n)$  at each  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ . Also, it is increasing in  $a_{-n}$ .*

Define a correspondence  $\tilde{\mathcal{B}}$  from the complete lattice  $\prod_{n \in N} \mathcal{M}(T_n, A_n)$  to itself so that

$$a' \in \tilde{\mathcal{B}}(a) \text{ if and only if } a'_n \in \tilde{\mathcal{B}}_n(a_{-n}) \text{ for any } n \in N. \quad (69)$$

Now Proposition 18 will make  $\tilde{\mathcal{B}}(a)$  a nonempty complete sublattice of  $\prod_{n \in N} \mathcal{M}(T_n, A_n)$  at every  $a \in \prod_{n \in N} \mathcal{M}(T_n, A_n)$  that is increasing in  $a$ . According to the discussion around (56), fixed points of  $\tilde{\mathcal{B}}$  will form pure and type-monotone equilibria of the special enterprising game  $\Gamma$  sense. Following the fixed point theorem of Zhou [53], which is a generalization of the classical result of Tarski [46], we have the following existence result.

**Theorem 5** *The set of  $\tilde{\mathcal{B}}$ 's fixed points,  $\tilde{\mathcal{E}} \equiv \{a \in \prod_{n \in N} \mathcal{M}(T_n, A_n) | a \in \tilde{\mathcal{B}}(a)\}$ , is a nonempty complete lattice. Thus,  $\Gamma$  has pure and monotone equilibria.*

In languages used earlier, Theorem 5 will result with

$$1_A \cap \mathcal{E}^a = 1_A \cap \mathcal{E}^d \supseteq 1_{\tilde{\mathcal{E}}} \equiv \{1_a \in 1_A | a \in \tilde{\mathcal{E}}\} \neq \emptyset. \quad (70)$$

As  $\tilde{\mathcal{E}}$  is a nonempty complete lattice, it has both the smallest and largest members. Let us denote them by  $\tilde{a}_*$  and  $\tilde{a}^*$ , respectively.

## 7.4 Monotone Comparative Statics

Let  $\Lambda$  be a partially ordered set, and let  $(\Gamma(\lambda) | \lambda \in \Lambda)$  be a family of special enterprising games finalized in Section 7.2. For  $\lambda \in \Lambda$ , suppose games  $\Gamma(\lambda)$  share a common set of players  $N$ , state space  $\tilde{\Omega}$ , type-space vector  $(T_n)_{n \in N}$ , and action-space vector  $(A_n)_{n \in N}$ . However, the utility functions  $(\tilde{u}_{n,t}(\lambda))_{n \in N, t \in T}$ , distributions  $(p_{n,t_n}(\lambda))_{n \in N, t_n \in T_n}$ , and prior sets  $(\tilde{\mathcal{Q}}_{n,t_n}(\lambda))_{n \in N, t_n \in T_n}$  are allowed to be  $\lambda$ -dependent.

We can define  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \nu | \lambda)$ ,  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu | \lambda)$ , and  $\tilde{s}_{n,t_n}(a_n, t_n, a_{-n} | \lambda)$ , respectively, using almost the same albeit  $\lambda$ -dependent (60), (59), and (58). We can then define  $\tilde{B}_{n,t_n}(a_{-n} | \lambda)$ ,  $\tilde{B}_n(a_{-n} | \lambda)$ , and  $\tilde{\mathcal{B}}(a | \lambda)$ , respectively, using almost the same albeit  $\lambda$ -dependent (56), (68), and (69). It is possible to predict how  $\tilde{\mathcal{B}}(\cdot | \lambda)$ 's extremal fixed points  $\tilde{a}_*(\lambda)$  and  $\tilde{a}^*(\lambda)$  would evolve with  $\lambda$  when the game  $\Gamma(\lambda)$ 's dependence on  $\lambda$  follows certain conditions. Let us list the latter in the following.

**Parametric Assumption 1** *For any  $n \in N$ ,  $t_n \in T_n$ ,  $t_{-n} \in T_{-n}$ , and  $a_{-n} \in A_{-n}$ , the payoff-utility function  $\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega} | \lambda)$  has increasing differences between  $(a_n, \tilde{\omega}) \in A_n \times \tilde{\Omega}$  and  $\lambda \in \Lambda$ .*

**Parametric Assumption 2** *For any  $n \in N$  and  $t_n \in T_n$ , the probability  $p_n^A \equiv (p_{n|t_{-n}}^A)_{t_{-n} \in T_{-n}}$  is invariant in  $\lambda \in \Lambda$ ; also, the probability  $p_{n,t_n}^B(\lambda) \equiv (p_{n,t_n|t_{-n}}^B(\lambda))_{t_{-n} \in T_{-n}}$  is monotone in  $\lambda \in \Lambda$  in the usual stochastic order.*

**Parametric Assumption 3** *For any  $n \in N$  and  $t_n \in T_n$ , the prior set  $\tilde{P}_{n,t_n,t_{-n}}^A(\lambda)$  is increasing in  $\lambda \in \Lambda$  for every  $t_{-n} \in T_{-n}$  and the prior set  $\tilde{P}_{n,t_n}^B(\lambda)$  is increasing in  $\lambda \in \Lambda$ .*

The increasing differences between  $a_n$  and  $\lambda$  in Parametric Assumption 1 and the monotonicity in  $\lambda$  of probabilities  $p_{n,t_n}^B(\lambda)$  in Parametric Assumption 2 are required for even the traditional game; see van Zandt and Vives [52]. When ambiguities on the external factors  $\tilde{\omega}$  are further involved, it should not be surprising that increasing differences between  $\tilde{\omega}$  and  $\lambda$  be postulated in Parametric Assumption 1 and the monotonicity in  $\lambda$  of prior sets be postulated in Parametric Assumption 3.

These assumptions will lead to the following intermediate results of the order-theoretic nature. The requirement in Parametric Assumption 2 that the probabilities  $p_n^A$  be invariant in  $\lambda$  is especially needed for the proof of Proposition 20.

**Proposition 19** *For  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu|\lambda)$  defined at the  $\lambda$ -dependent version of (60), it has increasing differences between  $(a_n, \mu) \in A_n \times \mathcal{P}(\tilde{\Omega})$  and  $\lambda \in \Lambda$ .*

**Proposition 20** *For  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu|\lambda)$  defined at the  $\lambda$ -dependent version of (59), it has increasing differences between  $a_n \in A_n$  and  $\lambda \in \Lambda$ . In addition, the function has increasing differences between  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  and  $\lambda \in \Lambda$  in scenario A and between  $\nu \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  and  $\lambda \in \Lambda$  in scenario B.*

**Proposition 21** *For  $\mathcal{Q}_{n,t_n}(\lambda)$  defined at the  $\lambda$ -dependent version of (61), regardless of the scenario that prevails, it is increasing in  $\lambda$  at each fixed  $n \in N$  and  $t_n \in T_n$ .*

**Proposition 22** *For  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n}|\lambda)$  defined at the  $\lambda$ -dependent version of (58), it has increasing differences between  $a_{n,t_n} \in A_n$  and  $\lambda \in \Lambda$  at each fixed  $n \in N$ ,  $t_n \in T_n$ , and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ .*

The key to the rest of the derivation is the monotonicity of the correspondence defined at (68). Ideas from Yang and Qi [51] can be borrowed in its proof.

**Proposition 23** *For the correspondence  $\tilde{\mathcal{B}}_n(a_{-n}|\lambda)$  defined at the  $\lambda$ -dependent (68), it is monotonically increasing in  $\lambda \in \Lambda$  at each fixed  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ .*

From Proposition 23 and the  $\lambda$ -dependent version of (69), we can immediately have the monotonicity of  $\tilde{\mathcal{B}}(a|\lambda)$  in  $\lambda$  at each fixed  $a \in \prod_{n \in N} \mathcal{M}(T_n, A_n)$ . Using Lemma 4 of Yang and Qi [51], a counterpart to Tarski's [46] monotone comparative statics result in Zhou's [53] setting, we can then achieve monotonicity of the extremal equilibria as  $\lambda$  varies.

**Theorem 6** *The enterprising game  $\Gamma(\lambda)$ 's smallest and largest pure and monotone equilibria,  $\tilde{a}_*(\lambda)$  and  $\tilde{a}^*(\lambda)$ , are both increasing in  $\lambda \in \Lambda$ .*

For scenario A, i.e., the one involving special type distributions but more general ambiguity attitudes on external factors, our general message about monotone equilibria that evolve monotonically over exogenous parameters is consistent with the one derived for the traditional expected-utility case. For the latter, see, e.g., van Zandt and Vives [52]. For scenario B, i.e., the one involving general type distributions but special ambiguity attitudes, barring some minutiae our results can even be thought of as generalizations of existing ones.

## 8 Potential Applications

### 8.1 Used-car Sales Involving Buyer Retaliation

We first describe an example befitting the general setup of Section 3.

Let the player set  $N$  be  $\{1, 2\}$ , so that player 1 is a used-car seller and player 2 a potential buyer. These two players have one used car to trade between them. Suppose the main decisions are not on the car's price which is close to being fixed, but rather on whether or not to proceed with the trade. Let  $T_1 = \{l, c\}$ , so that  $t_1 = l$  when the seller thinks the car is a lemon and  $t_1 = c$  when he thinks the car is a cherry. Here, even the seller is not absolutely sure of the car's quality. Let  $T_2 = \{v, e\}$ , so that  $t_2 = v$  when the buyer is vindictive and  $t_2 = e$  when she is easy-going. Suppose  $A_1 = A_2 = \{0, 1\}$ , in which 0 means no to selling or buying and 1 means yes to selling or buying.

Suppose  $\tilde{\omega} \in \tilde{\Omega} \equiv [0, 1]$  stands for the car's actual quality grade, the knowledge of which the seller has only some inkling of and the buyer has no idea about. If the buyer purchases the car, however, she will figure this out timely enough for some potential retaliatory actions in the form of bad online reviews, complaints to consumer protection agencies, or outright litigations. The initial profit the seller can earn by selling very likely depend on his own type  $t_1$ : with  $t_1 = l$  making a higher profit and  $t_1 = c$  making a lower one. On the other hand, to simplify matters, suppose that the consequences to both buyers and sellers of the buyer's potential retaliatory action will be determined solely by  $t_2 = v$  or  $e$  and  $\tilde{\omega} \in [0, 1]$ .

We let  $\Omega = T_1 \times T_2 \times \tilde{\Omega}$ , which is partitioned into the four pieces  $\Omega_{t_1, t_2} \equiv \{(t_1, t_2)\} \times \tilde{\Omega}$  for  $t_1 \in T_1 \equiv \{l, c\}$  and  $t_2 \in T_2 \equiv \{v, e\}$ . Note that every  $\Omega_{n, t_n} = T_m \times \tilde{\Omega}$  for any  $n = 1, 2$ ,  $t_n \in T_n$ , and  $m \neq n$ . The buyer's type certainly has nothing to do with the car's true quality. More strikingly, we have allowed the seller's estimate of the car, either  $l$  or  $c$ , to be potentially associable with any true quality grade within  $[0, 1]$ .

For the seller, let the payoff space  $R_{1, t_1}$  for either  $t_1 = l$  or  $c$  be  $[0, 1] \times [0, 1]$ , so that the first field contains the normalized profits he can earn by selling the car and the second field contains the normalized losses he will incur if the buyer takes retaliatory actions. His actual reward  $r_{1, t_1, t_2}(a_1, a_2, \omega)$  thus has two components. The first component will be 0 when either  $a_1$  or  $a_2$  is 0; it will record the  $t_2$ -dependent profit he can earn by selling the car when  $a_1 = a_2 = 1$ . The second component will be 0 when either there has been no trading or the buyer did not retaliate; also, it will record the loss to the seller when  $a_1 = a_2 = 1$  and  $t_2$  along with  $\tilde{\omega}$  dictate that retaliation did occur.

For the buyer, let the payoff space  $R_{2, t_2}$  for either  $t_2 = v$  or  $e$  be  $[0, 1] \times [0, 1] \times [0, 1]$ ,

so that the first field contains the potential qualities of the car she obtains, the second field contains the normalized payments she would make to the seller if trading is to occur, and the third field contains the normalized compensations she could receive if retaliatory actions were taken. Her actual reward  $r_{2,t_2,t_1}(a_2, a_1, \omega)$  will have three components. The first component will be 0 when there has been no trading while  $\tilde{\omega}$  otherwise. The second component will be 0 when there has been no trading, and it will record the price of the car otherwise. The third component will be 0 when either there has been no trading or the buyer did not retaliate; also, it will record the compensation received by the buyer when  $a_1 = a_2 = 1$  and  $t_2$  along with  $\tilde{\omega}$  dictate that retaliation did occur.

As long as we equip every  $(n, t_n)$ -pair with a preference relation  $\succ_{n,t_n}$  for the space  $(\mathcal{P}(R_{n,t_n}))^{\Omega_{n,t_n}}$  of payoff-distribution vectors, we will obtain a game  $\Gamma$  defined at the end of Section 3.1. The compactness-concerning Assumptions 1 to 3 are certainly satisfied. Suppose the payoff functions  $r_{n,t_n,t_{-n}}$  are continuous so that Assumption 4 is also true. Then, under Preference Assumption 1 which constitutes a merely mild requirement on the  $\succ_{n,t_n}$ 's, we will be able to use Theorem 1 to establish the existence of action-based equilibria. Both parties can decide on how to randomize their actions according to their own types.

When the preferences induce satisfaction functions  $s_{n,t_n}$  on the spaces  $(\mathcal{P}(R_{n,t_n}))^{\Omega_{n,t_n}}$ , we can instead test Satisfaction Assumption 1 to see if action-based equilibria can be guaranteed; see Corollary 1. The more interesting case is when satisfaction functions all take the form of either (3) or (4). If  $s_{1,l}$  takes the form of (3) along with (33), it probably means that the seller will be vigilant to potential lawsuits when he thinks the car is a lemon. When  $s_{2,e}$  takes the form of (4) along with (33), it probably indicates that the vindictive buyer can be opportunistic with her retaliatory options.

It is the member distributions  $\rho$  of the  $P_{n,t_n}$  sets associated with either (3) or (4) that will give us perceived associations between player types and other external factors. For instance, the notion that the seller has some knowledge about the car's true quality can be modeled by letting a majority of  $P_{1,l}$ 's members concentrate on low  $\tilde{\omega}$  values while letting a majority of  $P_{1,c}$ 's members concentrate on high  $\tilde{\omega}$  values. Meanwhile, the utility functions  $u_{n,t_n}$  in (33) will help rank players' multi-dimensional payoffs. For instance,  $u_{2,t_2}(r)$  will most likely increase with the first and third components of  $r \in R_{2,t_2}$ , and decrease with its second component. Also,  $u_{2,v}(r)$  will likely rise faster than  $u_{2,e}(r)$  in the first component of  $r$ , reflecting the vindictive type's heightened sensitivities to slights.

When all the prior sets  $P_{n,t_n}$  are singletons, we will get the traditional case as illustrated in Section 5.3. We just want to caution that the effective payoff  $v_{n,t_n,t_{-n}}(a_{n,t_n}, a_{-n,t_{-n}})$  defined

at (38) has now taken into account the buyer's potential retaliatory actions.

## 8.2 Auction with Ambiguity on Competitors' Assessments

Let us treat a single-item auction. Here, players are bidders and they only receive crude signals about the actual worths to them of the item being auctioned. For each player  $n \in N \equiv \{1, \dots, \bar{n}\}$ , let type space  $T_n \equiv \{1, \dots, \bar{t}_n\} \subseteq \mathfrak{R}$  denote the set of signals that player  $n$  can receive; also, let action space  $A_n \equiv [\underline{a}_n, \bar{a}_n] \subseteq \mathfrak{R}$  be the set of bids that player  $n$  can offer. We suppose the state space  $\tilde{\Omega} \equiv \prod_{n \in N} \tilde{\Omega}_n$  is made up of individual player-relevant compact real intervals  $\tilde{\Omega}_n \equiv [\underline{\omega}_n, \bar{\omega}_n]$ .

On the mechanism side, let  $\psi_n(a_n, a_{-n})$  be the chance that player  $n$  will get the item under bid profile  $(a_n, a_{-n})$ . Also, use  $\tau_n(a_n, a_{-n})$  for player  $n$ 's payment transfer to the auctioneer under bid profile  $(a_n, a_{-n})$ . In addition, let  $v_{n,t_n}(\tilde{\omega}_n)$  be the actual worth of the item to player  $n$  when he receives signal  $t_n$  and the external factor pertaining to him turns out to be  $\tilde{\omega}_n$ . For instance,  $v_{n,t_n}(\tilde{\omega}_n)$  might be  $t_n + \tilde{\omega}_n$ , so that  $t_n$  is a rough estimate and  $\tilde{\omega}_n$  is an additive adjustor. The payoff-utility for this problem can be taken as

$$\tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}) = \psi_n(a_n, a_{-n}) \cdot v_{n,t_n}(\tilde{\omega}_n) - \tau_n(a_n, a_{-n}). \quad (71)$$

This reflects that player  $n$  will earn the difference between the item's actual worth and his payment when he wins the bid, and he will still pay the transfer when he loses. Note the function is actually independent of  $t_{-n}$  and  $(\tilde{\omega}_m)_{m \neq n}$ .

Suppose there is no ambiguity on signals received by opponents. Following discussion around (57), we can use  $p_{n,t_n|t_{-n}}$  to stand for player  $n$ 's subjective probability of opponent-type profile being  $t_{-n}$  while his own type is  $t_n$ . Also, let us take (61) for the prior sets  $\mathcal{Q}_{n,t_n} \subseteq (\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  used in (57), with each  $\mathcal{K}_{n,t_n}$  equal to the entire  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  and each  $\tilde{P}_{n,t_n,t_{-n}} \subseteq \mathcal{P}(\prod_{n \in N} \tilde{\Omega}_n)$  serving as a set of priors on  $(\tilde{\omega}_n)_{n \in N}$ . This way, each player can have a different version of every player's item-worth distribution.

The otherwise traditional model of Milgrom and Weber [35] allowed bidders to be unsure of worths of the item being auctioned. Meanwhile, auction models of both Lo [29] and Bose, Ozdenoren, and Pape [7] incorporated ambiguity. However, bidders there are fully aware of the item's true worths to themselves, which also happen to be independent of other bidders' value assessments. In the current model, we let private message  $t_n$  represent only a rough estimate of the item's actual private worth, the latter of which depends also on the adjustor  $\tilde{\omega}_n$  through the  $v_{n,t_n}(\tilde{\omega}_n)$  function. Also, we allow ambiguities on the vector  $(\tilde{\omega}_n)_{n \in N}$  to be modeled through the prior sets  $\tilde{P}_{n,t_n,t_{-n}}$ .

The price we pay for more generality of the model is diminished understanding of the solution. So far, we can only use Corollary 3 to predict that action-based equilibria, likely of the mixed nature, would exist. Though stating the equivalence between pure equilibria of the action- and distribution-based varieties, Theorem 4 would not offer any answer on existence. As the  $\tilde{u}_{n,t_n,t_{-n}}$  function in (71) does not satisfy Monotonic Assumption 2, there is no hope to enlist the help of Theorem 5 to the latter task.

### 8.3 Competitive Pricing with Uncertain Demand

We now consider a situation that fits the description of Section 7. Players from the set  $N \equiv \{1, \dots, \bar{n}\}$  are firms engaged in price competition in a common market for their manufactured product items. Suppose it costs  $\underline{a}_n$  for firm  $n$  to manufacture a unit item. Also, let the firm's type  $t_n \in T_n \equiv \{1, \dots, \bar{t}_n\} \subseteq \mathfrak{R}$  stand for a factor of the demand that it is to face. The firm's action space  $A_n \equiv [\underline{a}_n, \bar{a}_n] \subseteq \mathfrak{R}$  denotes the range of prices that it can charge. Suppose the state space  $\tilde{\Omega} \equiv [0, \bar{\omega}]$  contains positive global additive factors to demands faced by all firms. We take the demand faced by firm  $n$  to be

$$\phi_{n,t_n}(a_n, a_{-n}, \tilde{\omega}) = \bar{b}_n - \bar{c}_n \cdot a_n + \sum_{m \neq n} \bar{d}_{nm} \cdot a_m + \bar{e}_n \cdot t_n + \bar{f}_n \cdot \tilde{\omega} + \bar{g}_n \cdot t_n \tilde{\omega}, \quad (72)$$

where  $\bar{b}_n$ ,  $\bar{c}_n$ ,  $(\bar{d}_{nm})_{m \neq n}$ ,  $\bar{e}_n$ ,  $\bar{f}_n$ , and  $\bar{g}_n$  are positive constants. Basically, demand to firm  $n$  will decline when the firm raises its price; but it will rise when competitors raise their prices. Moreover, both  $t_n$  and  $\tilde{\omega}$  serve as demand boosters, with the former being locally confined and the latter globally felt. The last term indicates that their effects may be compounded.

The profit that firm  $n$  can earn is therefore

$$\begin{aligned} \tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}) &= (a_n - \underline{a}_n) \cdot \phi_{n,t_n}(a_n, a_{-n}, \tilde{\omega}) \\ &= (a_n - \underline{a}_n) \cdot (\bar{b}_n - \bar{c}_n \cdot a_n + \sum_{m \neq n} \bar{d}_{nm} \cdot a_m + \bar{e}_n \cdot t_n + \bar{f}_n \cdot \tilde{\omega} + \bar{g}_n \cdot t_n \tilde{\omega}), \end{aligned} \quad (73)$$

which is independent of  $t_{-n}$ . More importantly, the function is increasing in  $\tilde{\omega}$ . So Monotonic Assumption 1 is satisfied. Taking derivatives, we obtain

$$\frac{\partial \tilde{u}_{n,t_n,t_{-n}}}{\partial a_n}(a_n, a_{-n}, \tilde{\omega}) = \underline{a}_n \bar{c}_n + \bar{b}_n - 2\bar{c}_n \cdot a_n + \sum_{m \neq n} \bar{d}_{nm} \cdot a_m + \bar{e}_n \cdot t_n + \bar{f}_n \cdot \tilde{\omega} + \bar{g}_n \cdot t_n \tilde{\omega}, \quad (74)$$

which is increasing in  $(t_n, t_{-n}, a_{-n}, \tilde{\omega})$ ; also,

$$\frac{\partial \tilde{u}_{n,t_n,t_{-n}}}{\partial \tilde{\omega}}(a_n, a_{-n}, \tilde{\omega}) = \bar{f}_n \cdot a_n - \underline{a}_n \bar{f}_n + \bar{g}_n \cdot (a_n - \underline{a}_n) \cdot t_n, \quad (75)$$

which is increasing in  $(t_n, t_{-n}, a_{-n})$ . Hence, Monotonic Assumption 2 is satisfied.



For local and global demand signals, suppose scenario A of Section 7.2 takes over. This means that players are unambiguous about their local signals but ambiguous about the global one. Also, Monotonic Assumption 3 is automatic. In particular, each firm  $n$  believes that other firms' types are distributed according to some  $p_n^A \equiv (p_{n|t_{-n}}^A)_{t_{-n} \in T_{-n}}$ , irrespective of its own type  $t_n$ ; moreover, there are prior sets  $\tilde{P}_{n,t_n,t_{-n}}^A$  so that the  $\mathcal{Q}_{n,t_n}$  used in (58) of its decision making process is defined through (66).

Now, suppose the  $P_{n,t_n,t_{-n}}^A$ 's are sublattices of  $\mathcal{P}(\tilde{\Omega})$  that also increase with  $t_n$ . The latter monotonicity connotes a certain positive correlation between a firm's local demand signal and the global one. Then, Monotonic Assumptions 4 and 5 will be satisfied. Thus, Theorem 5 can be used to predict that firms will be able to reach highest equilibrium pricing policies  $\tilde{a}_{n,t_n}^*$  that are increasing in their observed local signals  $t_n$ .

A partially ordered set  $\Lambda$  may provide parameters to the pricing game, so that the constants  $\bar{b}_n$ ,  $(\bar{d}_{nm})_{m \neq n}$ ,  $\bar{e}_n$ ,  $\bar{f}_n$ , and  $\bar{g}_n$  and prior sets  $\tilde{P}_{n,t_n,t_{-n}}^A$  are all functions of  $\lambda \in \Lambda$ . From (73), we can explain the monotonicity of those constants with respect to  $\lambda$  by an expanded demand base and demand's heightened sensitivities to other players' prices, as well as local and global signals. When this is the case, we will be able to learn from (74) and (75) the satisfaction of Parametric Assumption 1.

With the distributions  $p_n^A$  invariant in  $\lambda$ , Parametric Assumption 2 is automatic. Suppose further that a higher  $\lambda$  also reflects firms' bullish forecasts on the market, to the effect that the prior sets  $\tilde{P}_{n,t_n,t_{-n}}^A$  increase in  $\lambda$  as well. Then, Parametric Assumption 3 will be satisfied. The end result is that Theorem 6 can now be used to predict the increase of the highest monotone equilibrium  $\tilde{a}^*$  with respect to the changing  $\lambda$ . This result is quite anticipated, as bigger markets, more reactive demands, and brightened outlooks will embolden firms to price more aggressively.

## 9 Concluding Remarks

We allowed ambiguities on external factors to be treated in games involving incomplete information. For the two proposed behavioral-equilibrium concepts, we arrived to various results concerning their existence and mutual relationships. The enterprising game in which players are optimistic about the resolutions of their ambiguities delivered more concrete results. Not only are pure equilibria unified in such a game, but also their existence and monotone features are guaranteed when strategic complementarities are present.

Our framework could also enable the exploration of players' diverse risk attitudes. In

respect to space limitations, however, we have opted not to expand on this subject. More than providing normative answers to how participants should behave in situations involving both incomplete information and diverse ambiguity attitudes, some of our results might lead to explanations for phenomena already observed in real life. For instance, for auctions of works of art, offshore oilfields, electromagnetic spectra, etc., we speculate that uncertainties about the worths of items being auctioned and players' opportunistic attitudes toward the eventual resolutions of ambiguities might give extra impetus to upward movements of bidding prices. Hence, the winner's curse could be made even worse. Of course, we are in no position to offer any in-depth analysis here.

The model's confinement to finite type spaces could certainly hamper its applicabilities in some occasions. To deal with more general type spaces whilst still countenancing general ambiguity attitudes, it seems that information structures different from the current one revolving around the  $\Omega_{n,t_n}$  sets are warranted. This is besides the likely adoption of existing concepts such as distributional equilibrium taken up by say Milgrom and Weber [36]. In addition, issues concerning topologies on preference spaces, like those covered in Hildenbrand [23] and Klein and Thompson [27], will likely arise.

Equally importantly, we have not touched on players' ambiguities on opponents' strategies let alone their preferences. A completely different thinking should probably be adopted before one can embark on such ambitious undertakings. It will probably be a long journey to treat players' ambiguities on exogenous factors, opponents's behaviors, and opponents' preferences simultaneously. For example, works such as Epstein and Wang [18] and Di Tillio [13] on the emergence of natural type spaces will probably need expansions to accommodate the state space  $\Omega$ 's endogenization to accommodate players' type-dependent behaviors.

## Appendices

### A Proofs for Section 4

**Proof of Proposition 1:** By Assumptions 1, 2, and 4, we know that  $r_{n,t_n,t-n}$  is uniformly continuous. By a well known convergence result (Hildenbrand [23], D.I.(38)), this will lead to the continuity of each  $\pi_{n,t_n,t-n}^a$ , defined at (6) as a function from  $A_{n,t_n} \times \Delta_{-n,t-n} \times \Omega_{t_n,t-n}$  to  $\mathcal{P}(R_{n,t_n})$ , in  $(a_{n,t_n}, \omega)$  and also  $\delta_{-n,t-n}$ . Indeed, suppose  $\lim_{k \rightarrow +\infty} (a_{n,t_n}^k, \omega^k) = (a_{n,t_n}, \omega)$

and  $\lim_{k \rightarrow +\infty} \delta_{-n, t_{-n}}^k = \delta_{-n, t_{-n}}$ . Then, by (6) and  $r_{n, t_n, t_{-n}}$ 's continuity in  $(a_{n, t_n}, \omega)$  at an  $a_{-n, t_{-n}}$ -independent rate,  $\lim_{k \rightarrow +\infty} \pi_{n, t_n, t_{-n}}^a(a_{n, t_n}^k, \delta_{-n, t_{-n}}, \omega^k)$  equals

$$\begin{aligned} & \lim_{k \rightarrow +\infty} (\prod_{m \neq n} \delta_{m, t_m}) \cdot (r_{n, t_n, t_{-n}}(a_{n, t_n}^k, \cdot, \omega^k))^{-1} \\ &= (\prod_{m \neq n} \delta_{m, t_m}) \cdot (r_{n, t_n, t_{-n}}(a_{n, t_n}, \cdot, \omega))^{-1}, \end{aligned} \quad (\text{A.1})$$

which in turn equals  $\pi_{n, t_n, t_{-n}}^a(a_{n, t_n}, \delta_{-n, t_{-n}}, \omega)$ . Note  $\lim_{k \rightarrow +\infty} \delta_{-n, t_{-n}}^k = \delta_{-n, t_{-n}}$  will lead to

$$\lim_{k \rightarrow +\infty} \int_{A_{-n, t_{-n}}} y(a_{-n, t_{-n}}) \cdot \prod_{m \neq n} \delta_{m, t_m}^k(da_{m, t_m}) = \int_{A_{-n, t_{-n}}} y(a_{-n, t_{-n}}) \cdot \prod_{m \neq n} \delta_{m, t_m}(da_{m, t_m}), \quad (\text{A.2})$$

for any  $y \in \mathcal{C}(A_{-n, t_{-n}}, \mathbb{R})$ ; hence,  $\lim_{k \rightarrow +\infty} \prod_{m \neq n} \delta_{m, t_m}^k = \prod_{m \neq n} \delta_{m, t_m}$ . So similarly, by (6) and  $r_{n, t_n, t_{-n}}$ 's continuity in  $a_{-n, t_{-n}}$ , we will have  $\lim_{k \rightarrow +\infty} \pi_{n, t_n, t_{-n}}^a(a_{n, t_n}, \delta_{-n, t_{-n}}^k, \omega)$  equal to

$$\lim_{k \rightarrow +\infty} (\prod_{m \neq n} \delta_{m, t_m}^k) \cdot (r_{n, t_n, t_{-n}}(a_{n, t_n}, \cdot, \omega))^{-1} = (\prod_{m \neq n} \delta_{m, t_m}) \cdot (r_{n, t_n, t_{-n}}(a_{n, t_n}, \cdot, \omega))^{-1}. \quad (\text{A.3})$$

The following shows that  $\pi_{n, t_n, t_{-n}}^a$ 's continuity in  $(a_{n, t_n}, \omega)$  can be at a rate independent of the  $\delta_{-n, t_{-n}}$  present, just also because  $r_{n, t_n, t_{-n}}$ 's continuity in  $(a_{n, t_n}, \omega)$  is independent of  $a_{-n, t_{-n}}$ .

*Lemma* Let  $X$  and  $Y$  be separable metric spaces, and  $u$  and  $v$  be measurable functions from  $X$  to  $Y$ . Then, any  $\rho \in \mathcal{P}(X)$  satisfies

$$\psi_Y(\rho \cdot u^{-1}, \rho \cdot v^{-1}) \leq \sup_{x \in X} d_Y(u(x), v(x)), \quad (\text{A.4})$$

where the right-hand side is independent of the  $\rho$  involved.

*Proof.* Let  $\epsilon = \sup_{x \in X} d_Y(u(x), v(x))$ . There is nothing to prove if  $\epsilon = 0$  because then  $u = v$ . So suppose  $\epsilon > 0$ . For any  $Y' \in \mathcal{B}(Y)$ , we observe that

$$u^{-1}(Y') \subseteq v^{-1}((Y')^\epsilon). \quad (\text{A.5})$$

Thus,

$$(\rho \cdot u^{-1})(Y') \leq (\rho \cdot v^{-1})((Y')^\epsilon) < (\rho \cdot v^{-1})((Y')^\epsilon) + \epsilon. \quad (\text{A.6})$$

So by the definition of the Prokhorov metric  $\psi_Y$ , we have the desired inequality.

Combine the continuity in  $(a_{n, t_n}, \omega)$  at a  $\delta_{-n, t_{-n}}$ -independent rate and continuity in  $\delta_{-n, t_{-n}}$ , and we get  $\pi_{n, t_n, t_{-n}}^a$ 's continuity in  $(a_{n, t_n}, \delta_{-n, t_{-n}}, \omega)$ . We can similarly tackle  $\pi_{n, t_n, t_{-n}}^d$ 's continuity in  $(\delta_{n, t_n}, \delta_{-n, t_{-n}}, \omega)$ . ■

**Proof of Proposition 3:** We prove by contradiction. Suppose such a  $\succ_{n, t_n}$ -maximal  $\pi$  does not exist in  $\Pi'$ . Now for any  $\pi' \in \Pi'$ , define

$$L(\pi') = \{\pi \in \Pi' | \pi' \succ_{n, t_n} \pi\}. \quad (\text{A.7})$$

By the earlier hypothesis, every  $\pi \in \Pi'$  has a corresponding  $\pi'$  so that  $\pi \in L(\pi')$ . Thus,

$$\Pi' = \bigcup_{\pi' \in \Pi'} L(\pi'). \quad (\text{A.8})$$

In view of (20) and (A.7), each  $L(\pi')$  is a projection of  $H_{n,t_n} \equiv (\Pi_{n,t_n} \times \Pi_{n,t_n}) \setminus G_{n,t_n}$  to the first  $\Pi_{n,t_n}$ . Due to Preference Assumption 1, the set  $H_{n,t_n}$  is open. So must be every  $L(\pi')$ .

Since  $\Pi'$  is compact, from (A.8) we can infer that  $\Pi'$  has a finite subcover from among the  $L(\pi')$ 's. Pick a subcover with the smallest number of elements say  $k$ , involving open sets say  $L(\pi_1), \dots, L(\pi_k)$ . Suppose  $k \geq 2$ . By  $\succ_{n,t_n}$ 's irreflexibility and (A.7), we know

$$\pi_k \notin L(\pi_k). \quad (\text{A.9})$$

It must be the case that  $\pi_k \in L(\pi_l)$  for some  $l \leq k-1$ . Consider any  $\pi \in L(\pi_k)$ . Note that  $\pi_l \succ_{n,t_n} \pi_k$  and  $\pi_k \succ_{n,t_n} \pi$  by (A.7). By  $\succ_{n,t_n}$ 's transitivity, we must then have  $\pi_l \succ_{n,t_n} \pi$ ; that is,  $\pi \in L(\pi_l)$ . But this just means that  $L(\pi_k) \subseteq L(\pi_l)$ , a contradiction to the minimality of  $k$ . The only choice is  $k = 1$ . But this forces  $\pi_1 \in L(\pi_1)$ , an impossibility in view of (A.9). ■

**Proof of Proposition 4:** Suppose  $(a_{n,t_n}^k, \delta_{-n}^k) \in A_{n,t_n} \times \Delta_{-n}$  for  $k = 1, 2, \dots$  and  $(a_{n,t_n}, \delta_{-n}) \in A_{n,t_n} \times \Delta_{-n}$  are such that  $\lim_{k \rightarrow +\infty} a_{n,t_n}^k = a_{n,t_n}$ ,  $\lim_{k \rightarrow +\infty} \delta_{-n}^k = \delta_{-n}$ , and  $a_{n,t_n}^k \in \hat{A}_{n,t_n}^a(\delta_{-n}^k)$  for each  $k = 1, 2, \dots$ . We are to show that  $a_{n,t_n} \in \hat{A}_{n,t_n}^a(\delta_{-n})$  as well.

Let  $a'_{n,t_n}$  be arbitrarily chosen from  $A_{n,t_n}$ . By the membership of the  $a_{n,t_n}^k$ 's in the corresponding spaces  $\hat{A}_{n,t_n}^a(\delta_{-n}^k)$  and (14),

$$\pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}^k) \not\succ_{n,t_n} \pi_{n,t_n}^a(a_{n,t_n}^k, \delta_{-n}^k), \quad \forall k = 1, 2, \dots \quad (\text{A.10})$$

Proposition 2 and the first two conditions above will together lead to

$$\lim_{k \rightarrow +\infty} \pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}^k) = \pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}), \quad (\text{A.11})$$

$$\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}) = \lim_{k \rightarrow +\infty} \pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}^k) = \lim_{k \rightarrow +\infty} \pi_{n,t_n}^a(a_{n,t_n}^k, \delta_{-n}^k). \quad (\text{A.12})$$

Combining (A.10) to (A.12), as well as Preference Assumption 1, we get

$$\pi_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}) \not\succ_{n,t_n} \pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}). \quad (\text{A.13})$$

Due to (14) again and the arbitrariness of  $a'_{n,t_n} \in A_{n,t_n}$ , it follows that  $a_{n,t_n} \in \hat{A}_{n,t_n}^a(\delta_{-n})$ . ■

**Proof of Proposition 5:** Suppose  $\delta^k \equiv (\delta_m^k)_{m \in N} \equiv (\delta_{m,\tau_m}^k)_{m \in N, \tau_m \in T_m} \in \Delta \equiv \prod_{m \in N} \Delta_m \equiv \prod_{m \in N} \prod_{\tau_m \in T_m} \Delta_{m,\tau_m}$  for  $k = 1, 2, \dots$  and  $\delta \equiv (\delta_m)_{m \in N} \equiv (\delta_{m,\tau_m})_{m \in N, \tau_m \in T_m} \in \Delta$  are such

that  $\lim_{k \rightarrow +\infty} \delta_{n,t_n}^k = \delta_{n,t_n}$ ,  $\lim_{k \rightarrow +\infty} \delta_{-n}^k = \delta_{-n}$ , and  $\delta_{n,t_n}^k \in \hat{B}_{n,t_n}^a(\delta_{-n}^k)$  for each  $k = 1, 2, \dots$ . We are to show that  $\delta_{n,t_n} \in \hat{B}_{n,t_n}^a(\delta_{-n})$  as well. For this purpose, the closedness of  $\hat{A}_{n,t_n}^a(\cdot)$  as shown in Proposition 4 and the convergence of  $\delta_{-n}^k$  to  $\delta_{-n}$  will now lead to

$$\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots) \subseteq \hat{A}_{n,t_n}^a(\delta_{-n}), \quad (\text{A.14})$$

where  $\text{Ls}(\cdot)$  stands for a set sequence's topological limes superior; see Hildenbrand [23] (Section B.II). Let  $\epsilon > 0$  be given. Since  $\delta_{n,t_n}^k$  converges to  $\delta_{n,t_n}$ , as long as  $l$  is large enough,

$$\delta_{n,t_n}^l [(\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots))^\epsilon] \leq \delta_{n,t_n} [(\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots))^{2\epsilon}] + \epsilon. \quad (\text{A.15})$$

By the definition of  $\text{Ls}(\cdot)$  and  $A_{n,t_n}$ 's compactness, we have, when  $l$  is further large enough,

$$\hat{A}_{n,t_n}^a(\delta_{-n}^l) \subseteq (\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots))^\epsilon. \quad (\text{A.16})$$

Combining the above, we obtain

$$\begin{aligned} \delta_{n,t_n} [(\hat{A}_{n,t_n}^a(\delta_{-n}))^{2\epsilon}] &\geq \delta_{n,t_n} [(\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots))^{2\epsilon}] \\ &\geq \delta_{n,t_n}^l [(\text{Ls}(\hat{A}_{n,t_n}^a(\delta_{-n}^k) | k = 1, 2, \dots))^\epsilon] - \epsilon \geq \delta_{n,t_n}^l [\hat{A}_{n,t_n}^a(\delta_{-n}^l)] - \epsilon = 1 - \epsilon, \end{aligned} \quad (\text{A.17})$$

where the first inequality is due to (A.14), the second inequality is due to (A.15), the third inequality is due to (A.16), and the last equality comes from (15) and the membership of  $\delta_{n,t_n}^l$  in  $\hat{B}_{n,t_n}^a(\delta_{-n}^l)$ . For any  $k = 1, 2, \dots$ , this means that

$$\delta_{n,t_n} \left[ \bigcap_{l=k}^{+\infty} (\hat{A}_{n,t_n}^a(\delta_{-n}))^{1/l} \right] \geq 1 - \frac{1}{2k}. \quad (\text{A.18})$$

Since according to Proposition 4  $\hat{A}_{n,t_n}^a(\delta_{-n})$  is closed, and hence is equal to  $\bigcap_{l=k}^{+\infty} (\hat{A}_{n,t_n}^a(\delta_{-n}))^{1/l}$  for any  $k = 1, 2, \dots$ . So the above will result with  $\delta_{n,t_n}(\hat{A}_{n,t_n}^a(\delta_{-n})) = 1$ , translating into  $\delta_{n,t_n}$ 's membership in  $\hat{B}_{n,t_n}^a(\delta_{-n})$  by (15).  $\blacksquare$

## B Proofs for Section 6.1

**Proof of Lemma 1:** We first prove that  $\iota(y) \in \mathcal{P}(Z)$  at every  $y \in Y$ . Just because  $\kappa(x, y) \in \mathcal{P}(Z)$  at every  $x \in X$ , we can easily see from (43) that  $[\iota(y)](\emptyset) = 0$  and  $[\iota(y)](Z') + [\iota(y)](Z \setminus Z') = 1$  at every  $Z' \in \mathcal{B}(Z)$ . Given non-overlapping subsets  $Z^1, Z^2, \dots$  in  $\mathcal{B}(Z)$ , bounded convergence applied to (43) will also lead to

$$[\iota(y)]\left(\bigcup_{k=1}^{+\infty} Z^k\right) = \sum_{k=1}^{+\infty} [\iota(y)](Z^k). \quad (\text{B.1})$$

Thus,  $\iota(y)$  is a probability measure on the measurable space  $(Z, \mathcal{B}(Z))$ .

We next show that  $\iota$  is continuous from  $Y$  to  $\mathcal{P}(Z)$ . For sequence  $y^1, y^2, \dots$  that converges to  $y$  in  $Y$ , we know from (b) that  $\lim_{k \rightarrow +\infty} \kappa(x, y^k) = \kappa(x, y)$  at every  $x \in X$ . By the nature of the Prokhorov metric, this amounts to that, for every open subset  $Z'$  of  $Z$ ,

$$[\kappa(x, y)](Z') \leq \liminf_{k \rightarrow +\infty} [\kappa(x, y^k)](Z'). \quad (\text{B.2})$$

Now we can obtain

$$\begin{aligned} [\iota(y)](Z') &= \int_X [\kappa(x, y)](Z') \cdot \delta(dx) \leq \int_X \{\liminf_{k \rightarrow +\infty} [\kappa(x, y^k)](Z')\} \cdot \delta(dx) \\ &\leq \liminf_{k \rightarrow +\infty} \int_X [\kappa(x, y^k)](Z') \cdot \delta(dx) = \liminf_{k \rightarrow +\infty} [\iota(y^k)](Z'), \end{aligned} \quad (\text{B.3})$$

where the first equality is due to (43), the first inequality uses (B.2), the second inequality comes from Fatou's lemma, and the last equality is again due to (43). Since (B.3) applies to every open subset  $Z'$  of  $Z$ , it amounts to  $\lim_{k \rightarrow +\infty} \iota(y^k) = \iota(y)$ . Hence,  $\iota \in \mathcal{C}(Y, \mathcal{P}(Z))$ . ■

**Proof of Proposition 7:** We show that  $\pi_{n,t_n}^a(\delta_{-n})$  satisfies both (a) and (b) with  $X = A_{n,t_n}$ ,  $Y = \Omega_{n,t_n} \equiv \bigcup_{t_{-n} \in T_{-n}} \Omega_{t_n, t_{-n}}$ , and  $Z = R_{n,t_n}$ .

By Assumption 4, the payoff function  $r_{n,t_n,t_{-n}}(\cdot, \cdot, \omega)$  at every  $\omega \in \Omega_{t_n, t_{-n}}$  is continuous and hence measurable. So for any  $R'_{n,t_n} \in \mathcal{B}(R_{n,t_n})$ , the set  $(r_{n,t_n,t_{-n}}(\cdot, \cdot, \omega))^{-1}(R'_{n,t_n})$  is a member of  $\mathcal{B}(A_{n,t_n} \times A_{-n,t_{-n}})$ . Meanwhile, we can obtain from (6) and (7) that  $[\pi_{n,t_n,t_{-n}}^a(a_{n,t_n}, \delta_{-n,t_{-n}}, \omega)](R'_{n,t_n})$  is the integration of the indicator function of the measurable set  $(r_{n,t_n,t_{-n}}(\cdot, \cdot, \omega))^{-1}(R'_{n,t_n})$  over  $a_{-n,t_{-n}} \in A_{-n,t_{-n}}$  under the measure  $\prod_{m \neq n} \delta_{m,t_m}$ . So by Fubini's theorem,  $[\pi_{n,t_n,t_{-n}}^a(\cdot, \delta_{-n,t_{-n}}, \omega)](R'_{n,t_n})$  is a Borel-measurable mapping from  $A_{n,t_n}$  to  $[0, 1]$ . This means that (a) is satisfied by the vector  $\pi_{n,t_n}^a(\delta_{-n})$ . Also, Proposition 1 provides the continuity of  $\pi_{n,t_n,t_{-n}}^a(\cdot, \delta_{-n,t_{-n}}, \cdot)$  as a mapping from  $A_{n,t_n} \times \Omega_{t_n, t_{-n}}$  to  $\mathcal{P}(R_{n,t_n})$  at every  $t_{-n} \in T_{-n}$ . In view of the separation condition (19), we can obtain (b) as well. ■

## C Proofs for Section 6.2

**Proof of Proposition 8:** It will suffice to prove  $\hat{B}_{n,t_n}^d(\delta_{-n}) \subseteq \hat{B}_{n,t_n}^a(\delta_{-n})$  for every  $(n, t_n)$ -pair and  $\delta_{-n}$ . But by (47) and (48), the conclusion is immediate in view of the relation  $\mathcal{W}_{n,t_n}^d(\succ_{n,t_n}) \subseteq \mathcal{W}_{n,t_n}^a(\succ_{n,t_n})$ . ■

**Proof of Proposition 9:** It will suffice to prove  $\hat{B}_{n,t_n}^a(\delta_{-n}) \subseteq \hat{B}_{n,t_n}^d(\delta_{-n})$  for every  $(n, t_n)$ -pair and  $\delta_{-n}$ . But by (47) and (48), the conclusion is immediate in view of the relation

$$\mathcal{W}_{n,t_n}^a(\succ_{n,t_n}) \subseteq \mathcal{W}_{n,t_n}^d(\succ_{n,t_n}). \quad \blacksquare$$

**Proof of Proposition 10:** Suppose for continuous kernel  $\kappa_{n,t_n} \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$  and action distribution  $\delta_{n,t_n} \in \Delta_{n,t_n}$ , the value  $s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n}))$  is strictly below  $s_{n,t_n}(\kappa_{n,t_n}(a'_{n,t_n}))$  for some  $a'_{n,t_n} \in A_{n,t_n}$  at a  $\delta_{n,t_n}$ -positive set of  $a_{n,t_n}$ 's. Let

$$\bar{s}_{n,t_n} = \sup_{a_{n,t_n} \in A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})), \quad (\text{C.1})$$

which is finite as  $s_{n,t_n}$  is a continuous map on the compact space  $\Pi_{n,t_n}$ . Our hypothesis indicates that  $\delta_{n,t_n}(A'_{n,t_n}) > 0$  for  $A'_{n,t_n} = \{a_{n,t_n} \in A_{n,t_n} | s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) < \bar{s}_{n,t_n}\}$ . Note that  $A'_{n,t_n} = \bigcup_{l=1}^{+\infty} A_{n,t_n}^l$ , where

$$A_{n,t_n}^l = \{a_{n,t_n} \in A_{n,t_n} | s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) < \bar{s}_{n,t_n} - \frac{1}{l}\}, \quad \forall l = 1, 2, \dots \quad (\text{C.2})$$

So for some  $l$ , we have  $\delta_{n,t_n}(A_{n,t_n}^l) > 1/l > 0$ . Identify for this  $l$  an  $a_{n,t_n}^l \in A_{n,t_n}$  so that

$$s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n}^l)) \geq \bar{s}_{n,t_n} - \frac{1}{2l^2} \geq \bar{s}_{n,t_n} - \frac{1}{2l}. \quad (\text{C.3})$$

Now let  $\delta'_{n,t_n} \in \mathcal{P}(A_{n,t_n})$  be the Dirac measure on the point  $a_{n,t_n}^l$ . By this construction,

$$\begin{aligned} s_{n,t_n} \left( \left( \int_{A_{n,t_n}} \kappa_{n,t_n}(a'_{n,t_n}) \cdot \delta'_{n,t_n}(da'_{n,t_n}) \right) \right) &= s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n}^l)) \\ &\geq \int_{A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) \cdot \delta_{n,t_n}(da_{n,t_n}) + \delta_{n,t_n}(a_{n,t_n}^l)/(2l) - 1/(2l^2) \cdot (1 - 1/l) \\ &> \int_{A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) \cdot \delta_{n,t_n}(da_{n,t_n}), \end{aligned} \quad (\text{C.4})$$

which, by  $s_{n,t_n}$ 's strong convexity with respect to  $A_{n,t_n}$ , is greater than

$$s_{n,t_n} \left( \left( \int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n}) \right) \right). \quad (\text{C.5})$$

Therefore, the  $s_{n,t_n}$ -based  $\succ_{n,t_n}$  is individually prominent with respect to  $A_{n,t_n}$ .  $\blacksquare$

**Proof of Proposition 11:** Since  $s_{n,t_n}$ 's strong linearity implies its strong convexity, we know that  $\succ_{n,t_n}$  is individually prominent by Proposition 10.

For mixture preservation, suppose given continuous kernel  $\kappa_{n,t_n} \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$  and action distribution  $\delta_{n,t_n} \in \Delta_{n,t_n}$ , the value  $s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n}))$  is above  $s_{n,t_n}(\kappa_{n,t_n}(a'_{n,t_n}))$  for every  $a'_{n,t_n} \in A_{n,t_n}$  at  $\delta_{n,t_n}$ -almost every  $a_{n,t_n}$ . Then by the strong linearity of  $s_{n,t_n}$ ,

$$\begin{aligned} s_{n,t_n} \left( \int_{A_{n,t_n}} \kappa_{n,t_n}(a'_{n,t_n}) \cdot \delta'_{n,t_n}(da'_{n,t_n}) \right) &= \int_{A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a'_{n,t_n})) \cdot \delta'_{n,t_n}(da'_{n,t_n}) \\ &\leq \int_{A_{n,t_n}} s_{n,t_n}(\kappa_{n,t_n}(a_{n,t_n})) \cdot \delta_{n,t_n}(da_{n,t_n}) = s_{n,t_n} \left( \int_{A_{n,t_n}} \kappa_{n,t_n}(a_{n,t_n}) \cdot \delta_{n,t_n}(da_{n,t_n}) \right), \end{aligned} \quad (\text{C.6})$$

for any  $\delta'_{n,t_n} \in \Delta_{n,t_n}$ . Thus, the  $s_{n,t_n}$ -based  $\succ_{n,t_n}$  is mixture-preserving with respect to  $A_{n,t_n}$ . ■

**Proof of Proposition 12:** Under Assumptions 2 and 3, both the state space  $\Omega_{n,t_n}$  and payoff space  $R_{n,t_n}$  are compact and hence separable. Therefore,  $\Pi_{n,t_n} \equiv \mathcal{C}(\Omega_{n,t_n}, \mathcal{P}(R_{n,t_n}))$  equipped with the uniform metric based on the Prokhorov metric for  $R_{n,t_n}$  is homeomorphic to the infinite-dimensional Euclidean space  $\mathfrak{R}^\infty$ . The latter, being equipped with the metric induced from the  $l^\infty$ -norm, is a real topological vector space. Now, since  $\kappa_{n,t_n} \in \mathcal{K}(A_{n,t_n}, \Omega_{n,t_n}, R_{n,t_n})$  is continuous from  $A_{n,t_n}$  to  $\Pi_{n,t_n}$ , it can be treated as a continuous and hence measurable mapping from  $A_{n,t_n}$  to  $\mathfrak{R}^\infty$ . That is,  $\kappa_{n,t_n}$  is equivalent to a random variable say  $K_{n,t_n}$  with domain in the probability space  $(A_{n,t_n}, \mathcal{B}(A_{n,t_n}), \delta_{n,t_n})$  and range in the measurable space  $(\mathfrak{R}^\infty, \mathcal{B}(\mathfrak{R}^\infty))$ . Incidentally, (49) can be written as

$$s_{n,t_n}(\mathbb{E}[K_{n,t_n}]) \geq \mathbb{E}[s_{n,t_n}(K_{n,t_n})]. \quad (\text{C.7})$$

Then, using the general Jensen's inequality, we can deduce that ordinary concavity/convexity will lead to the so-called strong concavity/convexity. ■

## D Proofs for Section 6.3

**Proof of Theorem 3:** Suppose  $1_a \in \mathcal{E}^d$  for some  $a \equiv (a_{n,t_n})_{n \in N, t_n \in T_n}$  in the product action space  $\prod_{n \in N} \prod_{t_n \in T_n} A_{n,t_n}$ . Then, by (17) and (18),

$$\pi_{n,t_n}^d(\delta'_{n,t_n}, 1_{a_{-n}}) \not\succeq_{n,t_n} \pi_{n,t_n}^d(1_{a_{n,t_n}}, 1_{a_{-n}}), \quad \forall n \in N, t_n \in T_n, \delta'_{n,t_n} \in \Delta_{n,t_n}. \quad (\text{D.1})$$

Since any  $1_{A_{n,t_n}}$  is merely a subset of its corresponding  $\Delta_{n,t_n}$ , this results in

$$\pi_{n,t_n}^d(1_{a'_{n,t_n}}, 1_{a_{-n}}) \not\succeq_{n,t_n} \pi_{n,t_n}^d(1_{a_{n,t_n}}, 1_{a_{-n}}), \quad \forall n \in N, t_n \in T_n, a'_{n,t_n} \in A_{n,t_n}. \quad (\text{D.2})$$

Due to (51), this is the same as

$$\pi_{n,t_n}^a(a'_{n,t_n}, 1_{a_{-n}}) \not\succeq_{n,t_n} \pi_{n,t_n}^a(a_{n,t_n}, 1_{a_{-n}}), \quad \forall n \in N, t_n \in T_n, a'_{n,t_n} \in A_{n,t_n}. \quad (\text{D.3})$$

But (14) to (16) will give this the meaning of  $1_a \in \mathcal{E}^a$ . ■

**Proof of Proposition 13:** Due to (4) and (29),

$$s_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}) = \sup_{\rho \in P_{n,t_n}} s_{n,t_n}^0(\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}), \rho), \quad (\text{D.4})$$



where  $s_{n,t_n}^0$  is defined at (33). We have from (40) and (42) that

$$s_{n,t_n}^0(\pi_{n,t_n}^d(\delta_{n,t_n}, \delta_{-n}), \rho) = \int_{A_{n,t_n}} s_{n,t_n}^0(\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}), \rho) \cdot \delta_{n,t_n}(da_{n,t_n}). \quad (\text{D.5})$$

This means that  $s_{n,t_n}^0(\pi_{n,t_n}^d(\cdot, \delta_{-n}), \rho)$  is linear. With (D.4) showing it to be the supremum of linear functions, we can thus conclude that  $s_{n,t_n}^d(\cdot, \delta_{-n})$  is convex.

From (D.5), it is also clear that

$$s_{n,t_n}^0(\pi_{n,t_n}^d(1_{a_{n,t_n}}, \delta_{-n}), \rho) = s_{n,t_n}^0(\pi_{n,t_n}^a(a_{n,t_n}, \delta_{-n}), \rho). \quad (\text{D.6})$$

Hence, (4), (28), and (29) will together lead to the desired result.  $\blacksquare$

**Proof of Lemma 2:** The compactness of  $X$  will lead to that of  $\mathcal{P}(X)$ . Because  $f$  is continuous, some  $\xi^0 \in \mathcal{P}(X)$  will have achieved the supremum. The measure's support  $\text{supp}(\xi^0)$  is a closed subset of  $X$ . We are done if it is already a singleton. Suppose otherwise. Since  $X$  is compact in a finite-dimensional real Euclidean space say  $\mathbb{R}^k$ , it is bounded. Some closed rectangle  $Y^0$  with a finite total edge length say  $e^0 \equiv e_1^0 + \dots + e_k^0 > 0$  must have covered  $\text{supp}(\xi^0)$ . Without loss of generality, suppose  $e_1^0$  is the largest among all of  $Y^0$ 's edge lengths. Note that  $e_1^0 \geq e^0/k$ .

Consider the closed rectangle  $Y_L^0$  which takes the left half of  $Y^0$ 's first edge and the rest of its edges. Note that the closure of  $Y^0 \setminus Y_L^0$  is  $Y_R^0$ , the closed rectangle which takes the right half of  $Y^0$ 's first edge and the rest of its edges. For either the left- or right-half closed rectangle, the total edge length  $e^1$  is at most  $(2k-1)/(2k)$  times  $e^0$ . Suppose  $\xi^0(X \cap Y_L^0) = 0$ . Then,  $\text{supp}(\xi^0)$  is indeed covered by the smaller rectangle  $Y_R^0$ . Suppose  $\xi^0(X \cap (Y^0 \setminus Y_L^0)) = 0$ . Then,  $\text{supp}(\xi^0)$  is indeed covered by the smaller rectangle  $Y_L^0$ .

If neither is true, then we have both  $p_L^0 \equiv \xi^0(X \cap Y_L^0) > 0$  and  $p_R^0 \equiv \xi^0(X \cap (Y^0 \setminus Y_L^0)) = 1 - p_L^0 > 0$ . Consider members of  $\mathcal{P}(X)$ ,

$$\xi_L^0 \equiv \frac{1}{p_L^0} \cdot \xi^0|_{X \cap Y_L^0}, \quad \text{and} \quad \xi_R^0 \equiv \frac{1}{p_R^0} \cdot \xi^0|_{X \cap (Y^0 \setminus Y_L^0)}. \quad (\text{D.7})$$

Note that  $\text{supp}(\xi_L^0) \subseteq Y_L^0$  and  $\text{supp}(\xi_R^0) \subseteq Y_R^0$ . Also,

$$\xi^0 = p_L^0 \cdot \xi_L^0 + p_R^0 \cdot \xi_R^0. \quad (\text{D.8})$$

By the convexity of  $f$ ,

$$p_L^0 \cdot f(\xi_L^0) + p_R^0 \cdot f(\xi_R^0) \geq f(\xi^0). \quad (\text{D.9})$$

Since  $\xi^0$  has already achieved the supremum, both  $\xi_L^0$  and  $\xi_R^0$  must have too.

So no matter whichever one of the above three cases is present, we will be able to identify some supremum-reaching  $\xi^1 \in \mathcal{P}(X)$ , whose support  $\text{supp}(\xi^1)$  is covered by a closed rectangle  $Y^1$ , inside the original support-covering rectangle  $Y^0$  and with a total edge length  $e^1$  that is at most  $(2k - 1)/(2k)$  times the original  $e^0$ .

We can repeat the whole procedure from  $\xi^0$  to  $\xi^1$  incessantly. Then, we will get a sequence  $(\xi^n)_{n=0,1,\dots}$  of supremum-reaching distributions in  $\mathcal{P}(X)$ , whose supports are covered by increasingly nested closed rectangles  $Y^n$  with total edge lengths satisfying

$$e^{n+1} \leq \frac{2k-1}{2k} \cdot e^n, \quad \forall n = 0, 1, \dots \quad (\text{D.10})$$

Thus,  $\lim_{n \rightarrow +\infty} e^n = 0$ , and hence  $(X \cap Y^n)_{n=0,1,\dots}$  is a nested sequence of nonempty closed sets with shrinking dimensions in the compact set  $X$ . There must be one and only one member say  $x \in X$  inside all of the rectangles  $Y^n$ .

For any given  $\epsilon > 0$ , we know  $e^n$  will be smaller than it when  $n$  is large enough. For any closed subset  $F$  of  $\mathfrak{R}^k$ , let  $F^\epsilon$  be the set of all points that are within  $\epsilon$ -distance of  $F$ , where the distance between two points in  $\mathfrak{R}^k$  is measured through the  $l_1$ -norm. Because  $x$  is inside  $Y^n$ , whose total edge length  $e^n$  is below  $\epsilon$ , we can conclude that

$$x \notin X \cap F^\epsilon \implies X \cap F \cap Y^n = \emptyset, \quad (\text{D.11})$$

which will further lead to  $(X \cap F) \cap \text{supp}(\xi^n) = \emptyset$  because  $\text{supp}(\xi^n)$  is in  $Y^n$ . So depending on whether or not  $x \in X \cap F^\epsilon$ , we have either

$$1_x(X \cap F^\epsilon) + \epsilon = 1 + \epsilon > 1 \geq \xi^n(X \cap F), \quad (\text{D.12})$$

or

$$1_x(X \cap F^\epsilon) + \epsilon = \epsilon > 0 = \xi^n(X \cap F). \quad (\text{D.13})$$

But (D.12) and (D.13) together mean that

$$\psi_X(1_x, \xi^n) \leq \epsilon. \quad (\text{D.14})$$

That is, when measured by the Prokhorov metric  $\psi_X$  adopted to the distribution space  $\mathcal{P}(X)$ , the sequence  $(\xi^n)_{n=0,1,\dots}$  converges to  $1_x$ . By  $f$ 's continuity,

$$f(\xi^0) = f(\xi^1) = \dots = f(1_x), \quad (\text{D.15})$$

and hence the Dirac measure  $1_x$  has achieved the supremum. ■

**Proof of Theorem 4:** Combining Proposition 13 and (52), we obtain

$$\sup_{a'_{n,t_n} \in A_{n,t_n}} s_{n,t_n}^a(a'_{n,t_n}, \delta_{-n}) = \sup_{\delta'_{n,t_n} \in 1_{A_{n,t_n}}} s_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}) = \sup_{\delta'_{n,t_n} \in \Delta_{n,t_n}} s_{n,t_n}^d(\delta'_{n,t_n}, \delta_{-n}). \quad (\text{D.16})$$

Now suppose  $1_a \in \mathcal{E}^a$  for some  $a \equiv (a_{n,t_n})_{n \in N, t_n \in T_n}$  in the space  $\prod_{n \in N} \prod_{t_n \in T_n} A_{n,t_n}$ . Then, due to (30),

$$s_{n,t_n}^a(a_{n,t_n}, 1_{a-n}) = \sup_{a'_{n,t_n} \in A_{n,t_n}} s_{n,t_n}^a(a'_{n,t_n}, 1_{a-n}), \quad \forall n \in N, t_n \in T_n. \quad (\text{D.17})$$

By (D.16), this will lead to

$$s_{n,t_n}^a(a_{n,t_n}, 1_{a-n}) = \sup_{\delta'_{n,t_n} \in \Delta_{n,t_n}} s_{n,t_n}^d(\delta'_{n,t_n}, 1_{a-n}), \quad \forall n \in N, t_n \in T_n. \quad (\text{D.18})$$

But due to Proposition 13, we have further that

$$s_{n,t_n}^d(1_{a_{n,t_n}}, 1_{a-n}) = \sup_{\delta'_{n,t_n} \in \Delta_{n,t_n}} s_{n,t_n}^d(\delta'_{n,t_n}, 1_{a-n}), \quad \forall n \in N, t_n \in T_n. \quad (\text{D.19})$$

According to (31), this exactly means that  $1_a \in \mathcal{E}^d$ .

The above results with  $1_A \cap \mathcal{E}^a \subseteq 1_A \cap \mathcal{E}^d$ . We can reach our desired conclusion by combining this with either Corollary 4 or Theorem 3.  $\blacksquare$

## E Proofs for Section 7.3

**Proof of Proposition 14:** At any fixed  $n \in N$ ,  $t_n \in T_n$ , and  $t_{-n} \in T_{-n}$ , define  $q$  by

$$q(\tilde{\omega}) \equiv \tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}). \quad (\text{E.1})$$

The continuity of  $\tilde{u}_{n,t_n,t_{-n}}$  suggests that  $q$ , as a real-valued function defined on the compact  $\tilde{\Omega}$ , is lower bounded by some  $\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n})$ . Monotonic Assumption 1 also states that it is increasing. The former property indicates that  $\mu \cdot q^{-1}$  for any  $\mu \in \mathcal{P}(\tilde{\Omega})$  is a cumulative distribution function (cdf) say  $F$  on the real interval  $[\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), +\infty)$ . The latter property means that  $q^{-1}([\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x])$  is a lower set in the sense that  $\tilde{\omega}^1 \in q^{-1}([\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x])$  whenever  $\tilde{\omega}^2 \in q^{-1}([\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x])$  and  $\tilde{\omega}^1 \leq \tilde{\omega}^2$ . Thus, due to the partial order we have chosen for  $\mathcal{P}(\tilde{\Omega})$ , for any members  $\mu^1$  and  $\mu^2$ ,

$$\begin{aligned} & [(\mu^1 \vee \mu^2) \cdot q^{-1}](\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x) \\ &= (\mu^1 \cdot q^{-1})(\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x) \wedge (\mu^2 \cdot q^{-1})(\underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}), x). \end{aligned} \quad (\text{E.2})$$

The same applies to the opposite combination of “ $\vee$ ” and “ $\wedge$ ” as well. Then, for cdf’s  $F^1 \equiv \mu^1 \cdot q^{-1}$  and  $F^2 \equiv \mu^2 \cdot q^{-1}$ , we have

$$F^1 \wedge F^2 = (\mu^1 \vee \mu^2) \cdot q^{-1}, \quad F^1 \vee F^2 = (\mu^1 \wedge \mu^2) \cdot q^{-1}. \quad (\text{E.3})$$

Hence,

$$\begin{aligned} & \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot (\mu^1 \vee \mu^2)(d\tilde{\omega}) + \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot (\mu^1 \wedge \mu^2)(d\tilde{\omega}) \\ &= \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} x \cdot (F^1 \wedge F^2)(dx) + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} x \cdot (F^1 \vee F^2)(dx) \\ &= 2 \cdot \underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}) + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} (1 - F^1(x) \wedge F^2(x)) \cdot dx \\ &\quad + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} (1 - F^1(x) \vee F^2(x)) \cdot dx \\ &= 2 \cdot \underline{u}_{n,t_n,t_{-n}}(a_n, a_{-n}) + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} (1 - F^1(x)) \cdot dx \\ &\quad + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} (1 - F^2(x)) \cdot dx \\ &= \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} x \cdot F^1(dx) + \int_{\underline{u}_{n,t_n,t_{-n}}(a_n,a_{-n})}^{+\infty} x \cdot F^2(dx) \\ &= \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^1(d\tilde{\omega}) + \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^2(d\tilde{\omega}), \end{aligned} \quad (\text{E.4})$$

where the first and fifth equalities are achieved through changes of variables, the second and fourth equalities come from integrals by parts, and the third equality is due to the fact that

$$F^1(x) \wedge F^2(x) + F^1(x) \vee F^2(x) = F^1(x) + F^2(x). \quad (\text{E.5})$$

In view of (60) and (E.1), we have from (E.4) that  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \cdot)$  is both supermodular and submodular as a real-valued function defined on  $\mathcal{P}(\tilde{\Omega})$ .

Now suppose  $\mu \in \mathcal{P}(\tilde{\Omega})$  is fixed. Then, due to the average in (60) and the increasing differences stated in Monotonic Assumption 2,  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu)$  will have increasing differences between  $a_n \in A_n$  and  $(t_n, t_{-n}, a_{-n}) \in T_n \times T_{-n} \times A_{-n}$ . Next, suppose  $t_n^1, t_n^2 \in T_n$  satisfy  $t_n^1 \leq t_n^2$ ,  $t_{-n}^1, t_{-n}^2 \in T_{-n}$  satisfy  $t_{-n}^1 \leq t_{-n}^2$ ,  $a_n^1, a_n^2 \in A_n$  satisfy  $a_n^1 \leq a_n^2$ , and  $a_{-n}^1, a_{-n}^2 \in A_{-n}$  satisfy  $a_{-n}^1 \leq a_{-n}^2$ . Then, we note the increasing trend of  $q$  for

$$q(\tilde{\omega}) \equiv \tilde{u}_{n,t_n^2,t_{-n}^2}(a_n^2, a_{-n}^2, \tilde{\omega}) - \tilde{u}_{n,t_n^1,t_{-n}^1}(a_n^1, a_{-n}^1, \tilde{\omega}), \quad (\text{E.6})$$

due to the increasing differences between  $(t_n, t_{-n}, a_n, a_{-n}) \in T_n \times T_{-n} \times A_n \times A_{-n}$  and  $\tilde{\omega} \in \tilde{\Omega}$  as stated in Monotonic Assumption 2. Thus, for  $\mu^1, \mu^2 \in \mathcal{P}(\tilde{\Omega})$  satisfying  $\mu^1 \leq \mu^2$ ,

$$\int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^1(d\tilde{\omega}) \leq \int_{\tilde{\Omega}} q(\tilde{\omega}) \cdot \mu^2(d\tilde{\omega}). \quad (\text{E.7})$$

In view of (60) and (E.6), this will amount to the increasing differences that  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu)$  enjoys between  $(t_n, t_{-n}, a_n, a_{-n}) \in T_n \times T_{-n} \times A_n \times A_{-n}$  and  $\mu \in \mathcal{P}(\tilde{\Omega})$ .  $\blacksquare$

**Proof of Proposition 15:** First, the average in (59) and the simultaneous supermodularity and submodularity stated in Proposition 14 will lead to the simultaneous supermodularity and submodularity of  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  at every component  $\nu_{t_{-n}}$ . Due to the component-wise nature of the partial order assigned to  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , this also means that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  is both supermodular and submodular in  $\nu \equiv (\nu_{t_{-n}})_{t_{-n} \in T_{-n}} \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ .

Now suppose  $t_n \in T_n$  is fixed. Then, due to the average in (59) and the increasing differences stated in Proposition 14,  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  will have increasing differences between  $a_n \in A_n$  and  $(a_{-n}, \nu) \in \prod_{m \neq n} \mathcal{M}(T_m, A_m) \times \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , as well as between  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ .

Next, suppose  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  are fixed, while  $a_n^1, a_n^2 \in A_n$  satisfy  $a_n^1 \leq a_n^2$ . Define  $f \equiv (f_{t_n, t_{-n}})_{t_n \in T_n, t_{-n} \in T_{-n}}$  so that

$$f_{t_n, t_{-n}} \equiv \tilde{w}_{n,t_n,t_{-n}}(a_n^2, a_{-n,t_{-n}}, \nu_{t_{-n}}) - \tilde{w}_{n,t_n,t_{-n}}(a_n^1, a_{-n,t_{-n}}, \nu_{t_{-n}}). \quad (\text{E.8})$$

Due to the increasing differences between  $a_n \in A_n$  and  $(t_n, t_{-n}, a_{-n,t_{-n}}, \nu_{-n,t_{-n}}) \in T_n \times T_{-n} \times A_{-n} \times \mathcal{P}(\tilde{\Omega})$  as stated in Proposition 14, as well as the memberships of  $a_{-n}$  in  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu$  in  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , we know that  $f_{t_n, t_{-n}}$  is increasing in both  $t_n$  and  $t_{-n}$ . But by Monotonic Assumption 3, this will translate into

$$\sum_{t_{-n} \in T_{-n}} p_{n,t_n^1|t_{-n}} \cdot f_{t_n^1, t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n^1|t_{-n}} \cdot f_{t_n^2, t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n^2|t_{-n}} \cdot f_{t_n^2, t_{-n}}, \quad (\text{E.9})$$

whenever  $t_n^1 \leq t_n^2$ . In view of (59) and (E.8), this will amount to the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  enjoys between  $a_n \in A_n$  and  $t_n \in T_n$ .

Suppose  $a_n \in A_n$  and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  are fixed, while  $\nu^1, \nu^2 \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  satisfy  $\nu^1 \leq \nu^2$ . Define  $g \equiv (g_{t_n, t_{-n}})_{t_n \in T_n, t_{-n} \in T_{-n}}$  so that

$$g_{t_n, t_{-n}} \equiv \tilde{w}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^2) - \tilde{w}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^1). \quad (\text{E.10})$$

Due to the increasing differences between  $t_n \in T_n$  and  $\nu_{t_{-n}} \in \mathcal{P}(\tilde{\Omega})$  as stated in Proposition 14, we know that  $g_{t_n, t_{-n}}$  is increasing in  $t_n$ . But by averaging, this will translate into

$$\sum_{t_{-n} \in T_{-n}} p_{n|t_{-n}}^A \cdot g_{t_n^1, t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n|t_{-n}}^A \cdot g_{t_n^2, t_{-n}}, \quad (\text{E.11})$$

whenever  $t_n^1 \leq t_n^2$ . In view of (59) and (E.10), this will amount to the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu)$  enjoys between  $t_n \in T_n$  and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario A.

In addition, suppose  $a_n \in A_n$  and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  are fixed, while  $\nu^1, \nu^2 \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  satisfy  $\nu^1 \leq \nu^2$ . Define  $h \equiv (h_{t_n, t_{-n}})_{t_n \in T_n, t_{-n} \in T_{-n}}$  so that

$$h_{t_n, t_{-n}} \equiv \tilde{w}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^2) - \tilde{w}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^1). \quad (\text{E.12})$$

Due to  $\nu^1$  and  $\nu^2$ 's memberships in  $1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , neither  $\nu_{t_{-n}}^1$  nor  $\nu_{t_{-n}}^2$  depends on  $t_{-n}$ . So by the increasing differences between  $(t_n, t_{-n}, a_{-n, t_{-n}}) \in T_n \times T_{-n} \times A_{-n}$  and  $\nu_{t_{-n}} \in \mathcal{P}(\tilde{\Omega})$  as stated in Proposition 14, as well as the membership of  $a_{-n}$  in  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$ , we know that  $h_{t_n, t_{-n}}$  is increasing in both  $t_n$  and  $t_{-n}$ . But by Monotonic Assumption 3,

$$\sum_{t_{-n} \in T_{-n}} p_{n, t_n^1 | t_{-n}}^B \cdot h_{t_n^1, t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n, t_n^1 | t_{-n}}^B \cdot h_{t_n^2, t_{-n}} \leq \sum_{t_{-n} \in T_{-n}} p_{n, t_n^2 | t_{-n}}^B \cdot h_{t_n^2, t_{-n}}, \quad (\text{E.13})$$

whenever  $t_n^1 \leq t_n^2$ . In view of (59) and (E.12), this will amount to the increasing differences that  $\tilde{w}_{n, t_n}(a_n, a_{-n}, \nu)$  enjoys between  $t_n \in T_n$  and  $\nu \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario B. ■

**Proof of Proposition 16:** For scenario A, we first show that  $\mathcal{Q}_{n, t_n}^A$  as defined through (66) is a sublattice of  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ . Suppose  $\nu^1, \nu^2 \in \mathcal{Q}_{n, t_n}^A$ . Then, according to (66),  $\nu_{t_{-n}}^1, \nu_{t_{-n}}^2 \in \tilde{P}_{n, t_n, t_{-n}}^A$  for any  $t_{-n} \in T_{-n}$ . So by Monotonic Assumption 4,

$$\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2, \nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2 \in \tilde{P}_{n, t_n, t_{-n}}^A, \quad \forall t_{-n} \in T_{-n}. \quad (\text{E.14})$$

In addition, both  $\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2$  and  $\nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2$  will continue to be increasing in  $t_{-n}$  just because both  $\nu_{t_{-n}}^1$  and  $\nu_{t_{-n}}^2$  are. But by (66), this leads back to

$$\nu^1 \wedge \nu^2, \nu^1 \vee \nu^2 \in \mathcal{Q}_{n, t_n}^A. \quad (\text{E.15})$$

Hence,  $\mathcal{Q}_{n, t_n}^A$  is a sublattice of  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$ .

We next prove that  $\mathcal{Q}_{n, t_n}^A$  is nonempty. Since  $\tilde{P}_{n, t_n, t_{-n}}^A$  for each  $t_{-n} \in T_{-n}$  is nonempty, we can pick one  $\nu_{t_{-n}}$  from every  $\tilde{P}_{n, t_n, t_{-n}}^A$ . Let us go through every  $m \neq n$  in the order of  $m = 1, \dots, n-1, n+1, \dots, \bar{n}$ . Suppose we are at a particular  $m \neq n$ . Then, for every  $t_{-(n, m)} \in T_{-(n, m)} \equiv \prod_{l \neq n, m} T_l$ , we go through the procedure of

$$\nu_{t_m, t_{-(n, m)}} = \bigwedge_{\tau_m = t_m}^{\bar{t}_m} \nu'_{\tau_m, t_{-(n, m)}}, \quad \forall t_m = 1, 2, \dots, \bar{t}_m - 1, \quad (\text{E.16})$$

and

$$\nu_{t_m, t_{-(n, m)}} = \nu'_{t_m, t_{-(n, m)}}, \quad \forall t_m = 1, 2, \dots, \bar{t}_m - 1. \quad (\text{E.17})$$

We now show that the  $\nu$  assembled from all the  $\nu_{t_{-n}}$ 's after the procedure is a member of  $\mathcal{Q}_{n, t_n}^A$  as defined through (66). First, due to Monotonic Assumption 5, we can iteratively show that during the procedure,

$$\nu_{t_m, t_{-(n, m)}} \in \tilde{P}_{n, t_m, t_{-(n, m)}}^A, \quad \text{in the order of } t_m = \bar{t}_m, \bar{t}_m - 1, \dots, 1, \quad (\text{E.18})$$

for every  $t_{-(n,m)} \in T_{-(n,m)}$ . Second, after the procedure, for every  $m \neq n$  and every  $t_{-(n,m)} \in T_{-(n,m)}$ , the new  $\nu_{t_m, t_{-(n,m)}}$  is certainly increasing in  $t_m$ . For any  $t_{-n}^1, t_{-n}^2 \in T_{-n}$  satisfying  $t_{-n}^1 \leq t_{-n}^2$ , note that  $t_m^1 \leq t_m^2$  for any  $m \neq n$ . We can thus traverse from  $t_{-n}^1$  to  $t_{-n}^2$  in the order of  $t_{-n}^1 \equiv (t_1^1, t_{-(n,1)}^1), (t_1^1 + 1, t_{-(n,1)}^1), \dots, (t_1^2, t_{-(n,1)}^1) \equiv (t_1^2, t_2^1, t_{-(n,1,2)}^1), (t_1^2, t_2^1 + 1, t_{-(n,1,2)}^1), \dots, (t_{(1,2)}^2, t_{-(n,1,2)}^1) \equiv (t_{(1,2)}^2, t_3^1, t_{-(n,1,2,3)}^1), \dots, (t_{-(n,\bar{n})}^2, t_{\bar{n}}^2 - 1), (t_{-(n,\bar{n})}^2, t_{\bar{n}}^2) \equiv t_{-n}^2$ . Along the path, the  $t_{-n}$  encountered keeps rising. This implies that the associated  $\nu_{t_{-n}}$  would keep rising as well. Consequently, we can reach  $\nu_{t_{-n}^1} \leq \nu_{t_{-n}^2}$ . That is, the assembled  $\nu$  is a member of  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ . Taking both points together, we can see that  $\nu$  is a member of  $\mathcal{Q}_{n,t_n}^A$  as it is defined at (66). So  $\mathcal{Q}_{n,t_n}^A \neq \emptyset$ .

Finally, we verify the increasing trend of  $\mathcal{Q}_{n,t_n}^A$  in  $t_n$ . Suppose  $t_n^1, t_n^2 \in T_n$  satisfy  $t_n^1 \leq t_n^2$ ; also,  $\nu^1 \in \mathcal{Q}_{n,t_n^1}^A$  and  $\nu^2 \in \mathcal{Q}_{n,t_n^2}^A$ . Then, according to (66),  $\nu_{t_{-n}}^1 \in \tilde{P}_{n,t_n^1,t_{-n}}^A$  and  $\nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n^2,t_{-n}}^A$  for any  $t_{-n} \in T_{-n}$ , and both  $\nu_{t_{-n}}^1$  and  $\nu_{t_{-n}}^2$  are increasing in  $t_{-n}$ . So by Monotonic Assumption 5,

$$\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n^1,t_{-n}}^A, \quad \nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n^2,t_{-n}}^A, \quad \forall t_{-n} \in T_{-n}. \quad (\text{E.19})$$

Both  $\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2$  and  $\nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2$  are still increasing in  $t_{-n}$ . But by (66), this leads back to

$$\nu^1 \wedge \nu^2 \in \tilde{P}_{n,t_n^1}, \quad \nu^1 \vee \nu^2 \in \tilde{P}_{n,t_n^2}. \quad (\text{E.20})$$

Hence,  $\mathcal{Q}_{n,t_n}^A$  is increasing in  $t_n$ .

For scenario B, we know that  $\mathcal{Q}_{n,t_n}^B$  as defined through (67) is nonempty just because  $\tilde{P}_{n,t_n}^B$  is nonempty. It is a sublattice of  $(\mathcal{P}(\tilde{\Omega}))^{T_{-n}}$  just because, due to Monotonic Assumption 4,  $\tilde{P}_{n,t_n}^B$  is a sublattice of  $\mathcal{P}(\tilde{\Omega})$ . Also, it is increasing in  $t_n$  just because, due to Monotonic Assumption 5,  $\tilde{P}_{n,t_n}^B$  is increasing in  $t_n$ . ■

**Proof of Lemma 3:** Suppose  $x^1, x^2 \in X$  satisfy  $x^1 \leq x^2$  and  $y^1, y^2 \in Y$  satisfy  $y^1 \leq y^2$ . For any  $\epsilon > 0$ , we can choose  $z^{12} \in \tilde{Z}(y^2)$  so that

$$f(x^1, y^2, z^{12}) \geq \sup_{z \in \tilde{Z}(y^2)} f(x^1, y^2, z) - \epsilon = g(x^1, y^2) - \epsilon, \quad (\text{E.21})$$

and  $z^{21} \in \tilde{Z}(y^1)$  so that

$$f(x^2, y^1, z^{21}) \geq \sup_{z \in \tilde{Z}(y^1)} f(x^2, y^1, z) - \epsilon = g(x^2, y^1) - \epsilon. \quad (\text{E.22})$$

Since  $\tilde{Z}(y)$ 's are sublattices that increase with  $y$ ,

$$z^{12} \wedge z^{21} \in \tilde{Z}(y^1), \quad z^{12} \vee z^{21} \in \tilde{Z}(y^2). \quad (\text{E.23})$$

Now, by  $f$ 's increasing differences between  $x \in X$  and  $y \in Y$ ,

$$f(x^1, y^1, z^{12}) - f(x^2, y^1, z^{12}) + f(x^2, y^2, z^{12}) - f(x^1, y^2, z^{12}) \geq 0; \quad (\text{E.24})$$

by  $f$ 's increasing differences between  $x \in X$  and  $z \in Z$ ,

$$f(x^1, y^1, z^{21}) - f(x^2, y^1, z^{21}) - f(x^1, y^1, z^{12} \vee z^{21}) + f(x^2, y^1, z^{12} \vee z^{21}) \geq 0; \quad (\text{E.25})$$

by  $f$ 's supermodularity in  $z \in Z$ ,

$$f(x^1, y^1, z^{12} \wedge z^{21}) - f(x^1, y^1, z^{12}) - f(x^1, y^1, z^{21}) + f(x^1, y^1, z^{12} \vee z^{21}) \geq 0; \quad (\text{E.26})$$

in addition, by  $f$ 's increasing differences between  $y \in Y$  and  $z \in Z$ ,

$$f(x^2, y^1, z^{12}) - f(x^2, y^2, z^{12}) - f(x^2, y^1, z^{12} \vee z^{21}) + f(x^2, y^2, z^{12} \vee z^{21}) \geq 0. \quad (\text{E.27})$$

Adding up (E.24) to (E.27) together, we obtain

$$f(x^1, y^1, z^{12} \wedge z^{21}) - f(x^1, y^2, z^{12}) - f(x^2, y^1, z^{21}) + f(x^2, y^2, z^{12} \vee z^{21}) \geq 0. \quad (\text{E.28})$$

When this is combined with (E.21) to (E.23), we have

$$\begin{aligned} g(x^1, y^1) + g(x^2, y^2) &\geq f(x^1, y^1, z^{12} \wedge z^{21}) + f(x^2, y^2, z^{12} \vee z^{21}) \\ &\geq f(x^1, y^2, z^{12}) + f(x^2, y^1, z^{21}) \geq g(x^1, y^2) + g(x^2, y^1) - 2\epsilon. \end{aligned} \quad (\text{E.29})$$

Since  $\epsilon > 0$  can be made arbitrarily small, it follows that

$$g(x^1, y^1) + g(x^2, y^2) \geq g(x^1, y^2) + g(x^2, y^1). \quad (\text{E.30})$$

That is,  $g$  has increasing differences between  $x \in X$  and  $y \in Y$ . ■

**Proof of Proposition 17:** At any fixed  $n \in N$ , we can identify  $a_{n,t_n} \in A_n$  with  $x \in X$ ,  $(t_n, a_{-n}) \in T_n \times \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  with  $y \in Y$ , and  $\nu \in (\mathcal{P}(\tilde{\Omega}))^{T-n}$  with  $z \in Z$ . Also, we can identify  $\tilde{w}_{n,t_n}(a_{n,t_n}, a_{-n}, \nu)$  with  $f(x, y, z)$ ,  $\mathcal{Q}_{n,t_n}$  with  $\tilde{Z}(y)$ , and  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n})$  with  $g(x, y)$ . From Proposition 15, we know that (I), (II), and (V) of Lemma 3 are true. From Proposition 16, we know that (III) and (IV) of Lemma 3 are also true. Now (58) dictates that the relationship between  $f$ ,  $\tilde{Z}$ , and  $g$  in Lemma 3 also applies here. So by that lemma, we can derive that  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n})$  has increasing differences between  $a_{n,t_n} \in A_n$  and  $(t_n, a_{-n}) \in T_n \times \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ . ■



**Proof of Proposition 18:** The proof has similarities to that for Theorem 1 of Yang and Qi [51]. Define  $\tilde{b}_{n,t_n}(a_{-n})$  as player  $n$ 's highest best response to the given other-player monotone type-to-action profile  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  when his own type is  $t_n \in T_n$ :

$$\tilde{b}_{n,t_n}(a_{-n}) = \sup \tilde{B}_{n,t_n}(a_{-n}), \quad (\text{E.31})$$

where the latter set is defined at (56). The properties of  $\tilde{B}_{n,t_n}(a_{-n})$  will guarantee that  $\tilde{b}_{n,t_n}(a_{-n})$  is both well defined and monotone in  $(t_n, a_{-n})$ . So given  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ , the set  $\tilde{\mathcal{B}}_n(a_{-n})$  contains the element  $(\tilde{b}_{n,t_n}(a_{-n}))_{t_n \in T_n} \in \mathcal{M}(T_n, A_n)$  and hence is nonempty.

For an arbitrary nonempty subset  $B$  of  $\tilde{\mathcal{B}}_n(a_{-n})$ , we show that  $\sup B \in \tilde{\mathcal{B}}_n(a_{-n})$ . Let

$$B|_{t_n} = \{b' \in A_n | b' = b_{t_n} \text{ for some } b \equiv (b_{t_n})_{t_n \in T_n} \in B\}, \quad \forall t_n \in T_n. \quad (\text{E.32})$$

Due to (68),  $B|_{t_n}$  must be a subset of  $\tilde{B}_{n,t_n}(a_{-n})$ . But as the latter is a nonempty complete sublattice of  $A_n$ , we know that

$$\sup B|_{t_n} \in \tilde{B}_{n,t_n}(a_{-n}). \quad (\text{E.33})$$

Since the partial order on  $A_n^{T_n}$  is defined in the component-wise fashion, we have from (E.32)

$$\sup B = (\sup B|_{t_n})_{t_n \in T_n}. \quad (\text{E.34})$$

From the fact that  $B \subseteq \tilde{\mathcal{B}}_n(a_{-n}) \subseteq \mathcal{M}(T_n, A_n)$ , we also know that  $\sup B \in \mathcal{M}(T_n, A_n)$ . Therefore, according to (68),  $\sup B \in \tilde{\mathcal{B}}_n(a_{-n})$ . Symmetrically, we can also show that  $\inf B \in \tilde{\mathcal{B}}_n(a_{-n})$ . Thus the latter is a complete sublattice of  $\mathcal{M}(T_n, A_n)$ .

To show that  $\tilde{\mathcal{B}}_n$  is a monotone correspondence from  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$  to  $\mathcal{M}(A_n, T_n)$ , suppose  $a_{-n}^1, a_{-n}^2 \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  with  $a_{-n}^1 \leq a_{-n}^2$ . Since  $\tilde{B}_{n,t_n}$  is monotone in  $a_{-n}$ ,

$$\tilde{B}_{n,t_n}(a_{-n}^1) \leq \tilde{B}_{n,t_n}(a_{-n}^2), \quad \forall t_n \in T_n. \quad (\text{E.35})$$

For  $b^1 \equiv (b_{t_n}^1)_{t_n \in T_n} \in \tilde{\mathcal{B}}_n(a_{-n}^1)$  and  $b^2 \equiv (b_{t_n}^2)_{t_n \in T_n} \in \tilde{\mathcal{B}}_n(a_{-n}^2)$ , we have from (68) that

$$b_{t_n}^1 \in \tilde{B}_{n,t_n}(a_{-n}^1) \quad \text{and} \quad b_{t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n}^2), \quad \forall t_n \in T_n. \quad (\text{E.36})$$

But due to (E.35), we will have

$$b_{t_n}^1 \wedge b_{t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n}^1) \quad \text{and} \quad b_{t_n}^1 \vee b_{t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n}^2), \quad \forall t_n \in T_n. \quad (\text{E.37})$$

Note that  $b^1 \wedge b^2$  is merely  $(b_{t_n}^1 \wedge b_{t_n}^2)_{t_n \in T_n}$  and  $b^1 \vee b^2$  is merely  $(b_{t_n}^1 \vee b_{t_n}^2)_{t_n \in T_n}$ , and they are within  $\mathcal{M}(T_n, A_n)$  because  $b^1$  and  $b^2$  are. Hence, we have from (68) that

$$b^1 \wedge b^2 \in \tilde{\mathcal{B}}_n(a_{-n}^1) \quad \text{and} \quad b^1 \vee b^2 \in \tilde{\mathcal{B}}_n(a_{-n}^2). \quad (\text{E.38})$$

This will translate into the monotonicity of the correspondence  $\tilde{\mathcal{B}}_n$ . ■

## F Proofs for Section 7.4

**Proof of Proposition 19:** First, due to the average in the  $\lambda$ -dependent version of (60) and the increasing differences stated in Parametric Assumption 1,  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu|\lambda)$  will have increasing differences between  $a_n \in A_n$  and  $\lambda \in \Lambda$ . Next, suppose  $\lambda^1, \lambda^2 \in \Lambda$  satisfy  $\lambda^1 \leq \lambda^2$ . Then, we note the increasing trend of  $q$  for

$$q(\tilde{\omega}) \equiv \tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}|\lambda^2) - \tilde{u}_{n,t_n,t_{-n}}(a_n, a_{-n}, \tilde{\omega}|\lambda^1), \quad (\text{F.1})$$

due to the increasing differences between  $\tilde{\omega} \in \tilde{\Omega}$  and  $\lambda \in \Lambda$  as stated in Parametric Assumption 1. Thus, for  $\mu^1, \mu^2 \in \mathcal{P}(\tilde{\Omega})$  satisfying  $\mu^1 \leq \mu^2$ , we have the same relation as that stated in (E.7). In view of the  $\lambda$ -dependent version of (60) and (F.1), this will amount to the increasing differences that  $\tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n}, \mu|\lambda)$  enjoys between  $\mu \in \mathcal{P}(\tilde{\Omega})$  and  $\lambda \in \Lambda$ . ■

**Proof of Proposition 20:** Let  $n \in N$ ,  $t_n \in T_n$ , and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$  be fixed. First, suppose  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  is fixed, while  $a_n^1, a_n^2 \in A_n$  satisfy  $a_n^1 \leq a_n^2$ . Define  $f \equiv (f_{t_{-n}}(\lambda))_{t_{-n} \in T_{-n}, \lambda \in \Lambda}$  so that

$$f_{t_{-n}}(\lambda) \equiv \tilde{v}_{n,t_n,t_{-n}}(a_n^2, a_{-n,t_{-n}}, \nu_{t_{-n}}|\lambda) - \tilde{v}_{n,t_n,t_{-n}}(a_n^1, a_{-n,t_{-n}}, \nu_{t_{-n}}|\lambda). \quad (\text{F.2})$$

Due to the increasing differences between  $a_n \in A_n$  and  $(t_{-n}, a_{-n,t_{-n}}, \nu_{-n,t_{-n}}) \in T_{-n} \times A_{-n} \times \mathcal{P}(\tilde{\Omega})$  as stated in Proposition 14, as well as the memberships of  $a_{-n}$  in  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$  and  $\nu$  in  $\mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , we know that  $f_{t_{-n}}(\lambda)$  is increasing in  $t_{-n}$ . By the increasing differences between  $a_n \in A_n$  and  $\lambda \in \Lambda$  as stated in Proposition 19, we know that  $f_{t_{-n}}(\lambda)$  is increasing in  $\lambda$ . But by Parametric Assumption 2, these will translate into

$$\begin{aligned} \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}(\lambda^1) \cdot f_{t_{-n}}(\lambda^1) &\leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}(\lambda^1) \cdot f_{t_{-n}}(\lambda^2) \\ &\leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}(\lambda^2) \cdot f_{t_{-n}}(\lambda^2), \end{aligned} \quad (\text{F.3})$$

whenever  $\lambda^1 \leq \lambda^2$ . In view of the  $\lambda$ -dependent version of (59) and (F.2), this will amount to the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu|\lambda)$  enjoys between  $a_n \in A_n$  and  $\lambda \in \Lambda$ .

Now, suppose  $a_n \in A_n$  is fixed, while  $\nu^1, \nu^2 \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  satisfy  $\nu^1 \leq \nu^2$ . Define  $g \equiv (g_{t_{-n}}(\lambda))_{t_{-n} \in T_{-n}, \lambda \in \Lambda}$  so that

$$g_{t_{-n}}(\lambda) \equiv \tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^2|\lambda) - \tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^1|\lambda). \quad (\text{F.4})$$

Due to the increasing differences between  $\nu_{t_{-n}} \in \mathcal{P}(\tilde{\Omega})$  and  $\lambda \in \Lambda$  as stated in Proposition 19, we know that  $g_{t_{-n}}(\lambda)$  is increasing in  $\lambda$ . But by averaging over the probability  $p_n^A$  which is

invariant in  $\lambda$  according to Parametric Assumption 2, this will translate into

$$\sum_{t_{-n} \in T_{-n}} p_{n|t_{-n}}^A \cdot g_{t_{-n}}(\lambda^1) \leq \sum_{t_{-n} \in T_{-n}} p_{n|t_{-n}}^A \cdot g_{t_{-n}}(\lambda^2), \quad (\text{F.5})$$

whenever  $\lambda^1 \leq \lambda^2$ . In view of the  $\lambda$ -dependent version of (59) and (F.4), this will amount to the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu|\lambda)$  enjoys between  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  and  $\lambda \in \Lambda$  in scenario A.

In addition, suppose  $a_n \in A_n$  is fixed, while  $\nu^1, \nu^2 \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  satisfy  $\nu^1 \leq \nu^2$ . Define  $h \equiv (h_{t_{-n}}(\lambda))_{t_{-n} \in T_{-n}, \lambda \in \Lambda}$  so that

$$h_{t_{-n}}(\lambda) \equiv \tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^2|\lambda) - \tilde{v}_{n,t_n,t_{-n}}(a_n, a_{-n,t_{-n}}, \nu_{t_{-n}}^1|\lambda). \quad (\text{F.6})$$

Due to  $\nu^1$  and  $\nu^2$ 's memberships in  $1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$ , neither  $\nu_{t_{-n}}^1$  nor  $\nu_{t_{-n}}^2$  depends on  $t_{-n}$ . So by the increasing differences between  $(t_{-n}, a_{-n,t_{-n}}) \in T_{-n} \times A_{-n}$  and  $\nu_{t_{-n}} \in \mathcal{P}(\tilde{\Omega})$  as stated in Proposition 14, as well as the membership of  $a_{-n}$  in  $\prod_{m \neq n} \mathcal{M}(T_m, A_m)$ , we know that  $h_{t_{-n}}(\lambda)$  is increasing in  $t_{-n}$ . By the increasing differences between  $\nu_{t_{-n}} \in \mathcal{P}(\tilde{\Omega})$  and  $\lambda \in \Lambda$  as stated in Proposition 19, we know that  $h_{t_{-n}}(\lambda)$  is increasing in  $\lambda$ . But by Parametric Assumption 2,

$$\begin{aligned} \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}^B(\lambda^1) \cdot h_{t_{-n}}(\lambda^1) &\leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}^B(\lambda^1) \cdot h_{t_{-n}}(\lambda^2) \\ &\leq \sum_{t_{-n} \in T_{-n}} p_{n,t_n|t_{-n}}^B(\lambda^2) \cdot h_{t_{-n}}(\lambda^2), \end{aligned} \quad (\text{F.7})$$

whenever  $\lambda^1 \leq \lambda^2$ . In view of the  $\lambda$ -dependent version of (59) and (F.6), this will amount to the increasing differences that  $\tilde{w}_{n,t_n}(a_n, a_{-n}, \nu|\lambda)$  enjoys between  $\nu \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  and  $\lambda \in \Lambda$  in scenario B. ■

**Proof of Proposition 21:** For scenario A, suppose  $\lambda^1, \lambda^2 \in \Lambda$  satisfy  $\lambda^1 \leq \lambda^2$ ; also,  $\nu^1 \in \mathcal{Q}_{n,t_n}^A(\lambda^1)$  and  $\nu^2 \in \mathcal{Q}_{n,t_n}^A(\lambda^2)$ . Then, according to the  $\lambda$ -dependent version of (66),  $\nu_{t_{-n}}^1 \in \tilde{P}_{n,t_n,t_{-n}}^A(\lambda^1)$  and  $\nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n,t_{-n}}^A(\lambda^2)$  for any  $t_{-n} \in T_{-n}$ , and both  $\nu_{t_{-n}}^1$  and  $\nu_{t_{-n}}^2$  are increasing in  $t_{-n}$ . So by Parametric Assumption 3,

$$\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n,t_{-n}}^A(\lambda^1), \quad \nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2 \in \tilde{P}_{n,t_n,t_{-n}}^A(\lambda^2), \quad \forall t_{-n} \in T_{-n}. \quad (\text{F.8})$$

Both  $\nu_{t_{-n}}^1 \wedge \nu_{t_{-n}}^2$  and  $\nu_{t_{-n}}^1 \vee \nu_{t_{-n}}^2$  are still increasing in  $t_{-n}$ . But by the  $\lambda$ -dependent version of (66), this will lead back to

$$\nu^1 \wedge \nu^2 \in \mathcal{Q}_{n,t_n}^A(\lambda^1), \quad \nu^1 \vee \nu^2 \in \mathcal{Q}_{n,t_n}^A(\lambda^2). \quad (\text{F.9})$$

Hence,  $\mathcal{Q}_{n,t_n}^A(\lambda)$  is increasing in  $\lambda$ .

For scenario B, we know that  $\mathcal{Q}_{n,t_n}^B(\lambda)$  defined at the  $\lambda$ -dependent version of (67) is increasing in  $\lambda$  just because, due to Parametric Assumption 3,  $\tilde{P}_{n,t_n}^B(\lambda)$  is increasing in  $\lambda$ . ■

**Proof of Proposition 22:** At any fixed  $n \in N$ ,  $t_n \in T_n$ , and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ , we can identify  $a_{n,t_n} \in A_n$  with  $x \in X$ ,  $\lambda \in \Lambda$  with  $y \in Y$ , and  $\nu \in \mathcal{M}(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario A or  $\nu \in 1(T_{-n}, \mathcal{P}(\tilde{\Omega}))$  in scenario B with  $z \in Z$ . Also, we can identify  $\tilde{w}_{n,t_n}(a_{n,t_n}, a_{-n}, \nu | \lambda)$  with  $f(x, y, z)$ ,  $\mathcal{Q}_{n,t_n}(\lambda)$  with  $\tilde{Z}(y)$ , and  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n} | \lambda)$  with  $g(x, y)$ . From Propositions 15 and 20, we know that (I), (II), and (V) of Lemma 3 are true. We have the validity of Lemma 3's (III) from Proposition 16 and that of the lemma's (IV) from Proposition 21. Now the  $\lambda$ -dependent version of (58) dictates that the relationship between  $f$ ,  $\tilde{Z}$ , and  $g$  in Lemma 3 also applies here. So by that lemma, we can derive that  $\tilde{s}_{n,t_n}(a_{n,t_n}, a_{-n} | \lambda)$  has increasing differences between  $a_{n,t_n} \in A_n$  and  $\lambda \in \Lambda$ . ■

**Proof of Proposition 23:** The proof has similarities to that for Theorem 2 of Yang and Qi [51]. We first show that  $\tilde{B}_{n,t_n}(a_{-n} | \lambda)$  defined at the  $\lambda$ -dependent version of (56) is monotonically increasing in  $\lambda \in \Lambda$  at every  $n \in N$ ,  $t_n \in T_n$ , and  $a_{-n} \in \prod_{m \neq n} \mathcal{M}(T_m, A_m)$ . For that purpose, let  $\lambda^1, \lambda^2 \in \Lambda$  satisfying  $\lambda^1 \leq \lambda^2$ ,  $a_{n,t_n}^1 \in \tilde{B}_{n,t_n}(a_{-n} | \lambda^1)$ , and  $a_{n,t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n} | \lambda^2)$  be given. It can be checked that

$$\begin{aligned} 0 &\leq \tilde{s}_{n,t_n}(a_{n,t_n}^1, a_{-n} | \lambda^1) - \tilde{s}_{n,t_n}(a_{n,t_n}^1 \wedge a_{n,t_n}^2, a_{-n} | \lambda^1) \\ &\leq \tilde{s}_{n,t_n}(a_{n,t_n}^1, a_{-n} | \lambda^2) - \tilde{s}_{n,t_n}(a_{n,t_n}^1 \wedge a_{n,t_n}^2, a_{-n} | \lambda^2) \\ &= \tilde{s}_{n,t_n}(a_{n,t_n}^1 \vee a_{n,t_n}^2, a_{-n} | \lambda^2) - \tilde{s}_{n,t_n}(a_{n,t_n}^2, a_{-n} | \lambda^2) \leq 0, \end{aligned} \tag{F.10}$$

where the first inequality is due to the optimality of  $a_{n,t_n}^1$  for  $\tilde{s}_{n,t_n}(\cdot, a_{-n} | \lambda^1)$  as demanded by  $\tilde{B}_{n,t_n}(a_{-n} | \lambda^1)$ 's definition at the  $\lambda$ -dependent version of (56), the second inequality comes from Proposition 22, the only equality is attributable to the fact that  $A_n$  is totally ordered, and the last inequality is due to the optimality of  $a_{n,t_n}^2$  for  $\tilde{s}_{n,t_n}(\cdot, a_{-n} | \lambda^2)$  as demanded by  $\tilde{B}_{n,t_n}(a_{-n} | \lambda^2)$ 's definition at the  $\lambda$ -dependent version of (56). The only possibility is for all inequalities to be equalities. Thus, we must have  $a_{n,t_n}^1 \wedge a_{n,t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n} | \lambda^1)$  and  $a_{n,t_n}^1 \vee a_{n,t_n}^2 \in \tilde{B}_{n,t_n}(a_{-n} | \lambda^2)$ . Hence, the correspondence  $\tilde{B}_{n,t_n}(a_{-n} | \lambda)$  is increasing in  $\lambda$ .

In view of the definition of  $\tilde{\mathcal{B}}_n(a_{-n} | \lambda)$  from  $\tilde{B}_{n,t_n}(a_{-n} | \lambda)$  through the  $\lambda$ -dependent version of (68), it is clear that  $\tilde{\mathcal{B}}_n(a_{-n} | \lambda)$  will be increasing in  $\lambda$  as well. ■

## References

- [1] Ahn, D.S. 2007. Hierarchies of Ambiguous Beliefs. *Journal of Economic Theory*, **136**, pp. 286-301.
- [2] Allais, M. 1953. Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine. *Econometrica*, **21**, pp. 503-546.
- [3] Anscombe, F.J. and R.J. Aumann. 1963. A Definition of Subjective Probability. *The Annals of Mathematical Statistics*, **34**, pp. 199-205.
- [4] Aumann, R.J. 1976. Agreeing to Disagree. *Annals of Statistics*, **4**, pp. 1236-1239.
- [5] Azrieli, Y. and R. Teper. 2011. Uncertainty Aversion and Equilibrium Existence in Games with Incomplete Information. *Games and Economic Behavior*, **73**, pp. 310-317.
- [6] Bade, S. 2011. Ambiguous Act Equilibria. *Games and Economic Behavior*, **71**, pp. 246-260.
- [7] Bose, S. E. Ozdenoren, and A. Pape. 2006. Optimal Auctions with Ambiguity. *Theoretical Economics*, **1**, pp. 411-438.
- [8] Camerer C.F. and T.H. Ho. 1994. Violations of the Betweenness Axiom and Nonlinearity in Probability. *Journal of Risk and Uncertainty*, **8**, pp. 167-196.
- [9] Charness, G., E. Karni, and D. Levin. 2013. Ambiguity Attitudes and Social Interactions: An Experimental Investigation. *Journal of Risk and Uncertainty*, **46**, pp. 1-25.
- [10] Curley, S.P. and J.F. Yates. 1989. An Empirical Evaluation of Descriptive Models of Ambiguity Reactions in Choice Situations. *Journal of Mathematical Psychology*, **33**, pp. 397-427.
- [11] Debreu, G. 1964. Continuity Properties of Paretian Utility. *International Economic Review*, **5**, pp. 285-293.
- [12] Dempster, A.P. 1967. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Studies*, **38**, pp. 325-339.
- [13] Di Tillio. 2008. Subjective Expected Utility in Games. *Theoretical Economics*, **3**, pp. 287-323.

- [14] Dow, J. and S. Werlang. 1994. Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction. *Journal of Economic Theory*, **64**, pp. 305-324.
- [15] Eichberger, J. and D. Kelsey. 2000. Non-additive Beliefs and Strategic Equilibria. *Games and Economic Behavior*, **30**, pp. 183-215.
- [16] Ellsberg, D. 1961. Risk, Ambiguity and Savage Axioms. *Quarterly Journal of Economics*, **75**, pp. 643-669.
- [17] Epstein, L.G. 1997. Preference, Rationalizability, and Equilibrium. *Journal of Economic Theory*, **73**, pp. 1-29.
- [18] Epstein, L. and T. Wang. 1996. Beliefs about Beliefs without Probabilities. *Econometrica*, **64**, pp. 1343-1373.
- [19] Fagin, R. and J. Halpern. 1990. A New Approach to Updating Beliefs. *Proceedings of the 6th Conference on Uncertainty and AI*, pp. 317-325, Elsevier, New York.
- [20] Gilboa, I. and M. Marinacci. 2013. Ambiguity and the Bayesian Paradigm. In D. Acemoglu, M. Arellano, and E. Dekel (Eds.), *Advances in Economics and Econometrics: Theory and Applications, Tenth World Congress of the Econometric Society*. Cambridge University Press.
- [21] Gilboa, I. and D. Schmeidler. 1989. Maxmin Expected Utility with Non-unique Prior. *Journal of Mathematical Economics*, **18**, pp. 141-153.
- [22] Harsanyi, J.C. 1967-68. Games with Incomplete Information Played by Bayesian Players. *Management Science*, **14**, pp. 159-182, 320-334, 486-502.
- [23] Hildenbrand, W. 1974. *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton, NJ.
- [24] Kajii, A. and T. Ui. 2005. Incomplete Information Games with Multiple Priors. *Japanese Economic Review*, **56**, pp. 332-351.
- [25] Klibanoff, P. 1996. Uncertainty, Decision, and Normal Form Games. Mimeo, Northwestern University.
- [26] Khan, M.A. and Y.N. Sun. 1990. On a Reformulation of Cournot-Nash Equilibria. *Journal of Mathematical Analysis and Applications*, **146**, pp. 442-460.

- [27] Klein, E. and A.C. Thompson. 1984. *Theory of Correspondences*. John Wiley & Sons, New York.
- [28] Lo, K.-C. 1996. Equilibrium in Beliefs under Uncertainty. *Journal of Economic Theory*, **71**, pp. 443-484.
- [29] Lo, K.-C. 1998. Sealed Bid Auctions with Uncertain Averse Bidders. *Economic Theory*, **12**, pp. 1-20.
- [30] Marinacci, M. 2000. Ambiguous Games. *Games and Economic Behavior*, **31**, pp. 191-219.
- [31] Mas-Colell, A. 1974. An Equilibrium Existence Theorem without Complete or Transitive Preferences. *Journal of Mathematical Economics*, **1**, pp. 237-246.
- [32] Mertens, J.F. and S. Zamir. 1985. Formulation of Bayesian Analysis for Games and Incomplete Information. *International Journal of Game Theory*, **14**, pp. 1-29.
- [33] Milgrom, P.R. and J. Roberts. 1990. Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities. *Econometrica*, **58**, pp. 1255-1277.
- [34] Milgrom, P.R. and C. Shannon. 1994. Monotone Comparative Statics. *Econometrica*, **62**, pp. 157-180.
- [35] Milgrom, P.R. and R.J. Weber. 1982. A Theory of Auctions and Competitive Bidding. *Econometrica*, **50**, pp. 1089-1122.
- [36] Milgrom, P.R. and R.J. Weber. 1985. Distributional Strategies for Games with Incomplete Information. *Mathematics of Operations Research*, **10**, pp. 619-632.
- [37] Munkres, J.R. 2000. *Topology, 2nd Edition*, Prentice Hall, New Jersey.
- [38] Nash, J.F. 1950. Equilibrium Points in  $n$ -person Games. *Proceedings of the National Academy of Sciences*, **36**, pp. 48-49.
- [39] Nash, J.F. 1951. Non-cooperative Games. *Annals of Mathematics*, **54**, pp. 286-295.
- [40] von Neumann, J. and O. Morgenstern. 1944. *Theory of Games and Economic Behaviour*. Princeton University Press, Princeton, NJ.
- [41] Savage, L.J. 1972. *The Foundation of Statistics, 2nd Revised Edition*. Dover Publications, New York.

- [42] Schmeidler, D. 1969. Competitive Equilibria in Markets with a Continuum of Traders and Incomplete Preferences. *Econometrica*, **37**, pp. 578-585.
- [43] Schmeidler, D. 1989. Subjective Probability and Expected Utility without Additivity. *Econometrica*, **57**, pp. 571-587.
- [44] Shafer, W. and H. Sonnenschein. 1975. Equilibrium in Abstract Economies without Ordered Preferences. *Journal of Mathematical Economics*, **2**, pp. 345-348.
- [45] Shaked, M. and J.G. Shanthikumar. 2007. *Stochastic Orders*. Springer, Berlin.
- [46] Tarski, A. 1955. A Lattice-theoretical Fixpoint Theorem and Its Applications. *Pacific Journal of Mathematics*, **5**, pp. 285-309.
- [47] Topkis, D.M. 1979. Equilibrium Points in Nonzero-sum  $n$ -person Submodular Games. *SIAM Journal on Control and Optimization*, **17**, pp. 773-787.
- [48] Topkis, D.M. 1998. *Supermodularity and Complementarity*. Princeton University Press, Princeton, NJ.
- [49] Vives, X. 1990. Nash Equilibrium with Strategic Complementarities. *Journal of Mathematical Economics*, **19**, pp. 305-321.
- [50] Wu, G. and R. Gonzalez. 1996. Curvature of the Probability Weighing Function. *Management Science*, **42**, pp. 1676-1690.
- [51] Yang, J. and X. Qi. 2013. The Nonatomic Supermodular Game. *Games and Economic Behavior*, **82**, pp. 609-620.
- [52] van Zandt, T. and X. Vives. 2007. Monotone Equilibrium in Bayesian Games of Strategic Complementarities. *Journal of Economic Theory*, **134**, pp. 330-360.
- [53] Zhou, L. 1994. The Set of Nash Equilibria of a Supermodular Game is a Complete Lattice. *Games and Economic Behavior*, **7**, pp. 295-300.